MATH 212

Lab: Numerical Integration Techniques

Due: end of class on the day assigned (see Course Schedule in Syllabus)

Estimated Time: 1-2 hours

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Directions: Write your answers next to the "Ans" prompts. Submit to Google classroom by clicking the "Turn In" button by the end of class.

Grading: There are 25 points possible in this lab, with points noted next to the "Ans" prompt for each question you respond to.

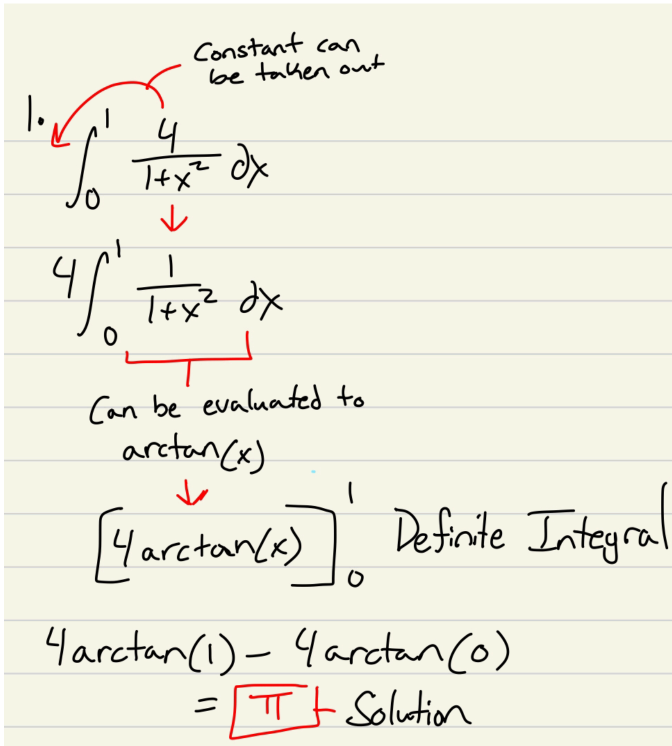
- See the Rubric for Labs in the Google classroom Lab assignment!
- You may discuss responses as a group, but each individual must refine the work and submit their own version of the lab, containing their own spreadsheet solutions, explanations and their own handwritten work for each part.
- The explanations and mathematical work must give evidence of mastery - this is more likely with complete sentences, few or no errors, and clear concise solutions including some words rather than only numbers or symbols.
- Revisions are accepted, up to 2 weeks after receipt of your grade; see syllabus.

Overview: In this lab we explore how we can use familiar integral estimates via Riemann sums (left, right, and middle sums) to generate even better estimates.

1. One of the reasons numerical integration techniques are important is that many integrals cannot be evaluated using the Fundamental Theorem of Calculus since we cannot find an elementary antiderivative. For example, we have discussed this for the integral $\int_0^1 e^{-x^2}dx$.
   However, to measure the accuracy of various approximations, it's helpful to start with a "baseline" integral that we are able to evaluate exactly for comparison. We will use the following integral for that purpose in the first part of this lab:

$$\int_0^1 \frac{4}{1+x^2}dx$$

Use the First FTC to evaluate this integral exactly. Clearly show your work and thinking using proper notation, and express the resulting value without a decimal approximation. You can do this by inserting equations, or on paper by hand and using “Insert > Image”.

(2) **Ans**: 

2. Let’s now estimate the value of the definite integral $\int_0^1 \frac{4}{1+x^2}dx$ using left, right, and middle Riemann sums, with some help from Desmos online graphing calculator.

   To start, make your own copy of this spreadsheet, and do the following things:

   a. enter the exact value you found in #1 in cell A2; if you need to enter π in some way, use “=PI()”. Copy cell A2 into A3, A4, and A5 as well.
   b. for the four Δx values in B2, B3, B4, and B5, determine the *n* value that corresponds to that Δx, and record your *n* values in C2, C3, C4, and C5. Recall, *n* is the number of rectangles you will use in your estimate and is calculated by dividing up the whole interval into *n* subintervals of width Δx: $n = (b - a)/\Delta x$.
   c. use Desmos (as shown in class before the lab and in the warmup) to find the four needed values of L_n and R_n (the Left and Right Riemann sums) and record those values in columns D and E. Record 8 digits of accuracy after each decimal place in the results from Desmos.

d. Finally, with no more help from Desmos, determine the values of T_n (the trapezoid rule) in column F.

Write one sentence below to explain how you found the T_n values, and then copy your rows 1-5 and columns A-F of your spreadsheet into the space below.

(5) **Ans**: I found the T_n values by following the trapezoidal formula therefore I added the Left and Right riemann sums then dividing it by two.

exact integral	delta x	n	L_n (via Desmos)	R_n (via Desmos)	T_n
3.141592654	0.1	10.000000000	3.23992598	3.03992598	3.13992598
3.141592654	0.05	20.000000000	3.19117598	3.09117598	3.14117598
3.141592654	0.025	40.000000000	3.16648848	3.11648848	3.14148848
3.141592654	0.0125	80.000000000	3.15406661	3.12906661	3.14156661

3. Next, use Desmos similarly to find the results of the Midpoint Rule for the four relevant *n* values and record your results in column G. Record 8 digits of accuracy after each decimal place in the results from Desmos.

Now we are going to focus our attention on the errors in the Trapezoid and Midpoint Rules.

The **Trapezoid Error** is defined to be $E_{T,n} = I - T_n$, where *I* is the exact value of the definite integral. Similarly, the **Midpoint Error** is defined to be $E_{M,n} = I - M_n$, where *I* is the exact value of the definite integral.

In your spreadsheet, enter formulas in columns H and I so that they measure the respective errors in the Trapezoid and Midpoint Rules. Then, copy rows 1-5 and columns A through I from your spreadsheet into the space below and write at least 2-3 sentences about what patterns you observe in the Trapezoid Error and Midpoint Error. There are several important things you can say.

(5) **Ans:** In the trapezoid error column we can observe that it overestimates the integral by being positive and above 0. While the Midpoint Error column seems to slightly underestimate the integral by being negative and below 0.

exact integral	delta x	n	L_n (via Desmos)	R_n (via Desmos)	T_n	M_n (via Desmos)	error(T_n)	error(M_n)

3.1415926 54	0.1	10.000000 000	3.2399259 8	3.0399259 8	3.1399259 8	3.1424259 8	0.0016666 7	-0.000833 33
3.1415926 54	0.05	20.000000 000	3.1911759 8	3.0911759 8	3.1411759 8	3.1418009 8	0.0004166 7	-0.000208 33
3.1415926 54	0.025	40.000000 000	3.1664884 8	3.1164884 8	3.1414884 8	3.1416447 3	0.0001041 7	-0.000052 08
3.1415926 54	0.0125	80.000000 000	3.1540666 1	3.1290666 1	3.1415666 1	3.1416056 7	0.0000260 4	-0.000013 02

4.

5. An important idea in "numerical analysis" (the formal study of implementing algorithms to solve mathematical problems - see MATH 328) is how to use as few computations as possible to gain highly accurate approximations. Indeed, one option for estimating an integral is simply to use a very large *n* value; but there are problems of both efficiency and accuracy that arise when *n* gets too big.

   Instead, we want to leverage existing computations in order to do even better. In #3, you should have observed key patterns in the errors of T_n and M_n - one regarding their signs, and another regarding their relative sizes. We are now going to use that observation to define a new rule with a weighted average of T_n and M_n: let

$$S_n = \frac{2M_n + T_n}{3} = \frac{2}{3}M_n + \frac{1}{3}T_n$$

   Explain why you expect the result of S_n to have much smaller error than either T_n or M_n.

   (2) **Ans:** From what I had observed, I would expect the result of S_n to have a much smaller error than T_n or M_n because of the opposing signs and produce a closer number to 0.

   Now, implement the new rule for S_n in column J of your spreadsheet and, in addition, use column K to compute the error of the S_n rule. Paste your full spreadsheet in the space below and, in addition, write at least two sentences that summarize what you observe about the error of this new rule (this is SImpson's rule).

   (5) **Ans:** With the implementation of Simpson's rule the error produced is much closer to 0 than T_n or M_n. The decimal places for S_n are more accurate to the exact integral.

exact integral	delta x	n	L_n (via Desmos)	R_n (via Desmos)	T_n	M_n (via Desmos)	error(T_n)	error(M_n)	S_n	error(S_n)
3.141592654	0.1	10.000000000	3.23992598	3.03992598	3.13992598	3.14242598	0.00166667	-0.00083333	3.141592647	0.000000006923126517
3.141592654	0.05	20.000000000	3.19117598	3.09117598	3.14117598	3.14180098	0.00041667	-0.00020833	3.141592647	0.000000006923126517
3.141592654	0.025	40.000000000	3.16648848	3.11648848	3.14148848	3.14164473	0.00010417	-0.00005208	3.141592647	0.000000006923126517
3.141592654	0.0125	80.000000000	3.15406661	3.12906661	3.14156661	3.14160567	0.00002604	-0.00001302	3.14159265	0.000000003589793351

6.

7. Suppose we have the following data for a braking car:

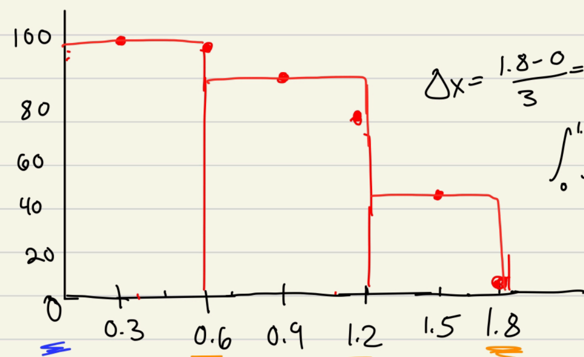
t (sec)	0	0.3	0.6	0.9	1.2	1.5	1.8
v(*t*) (ft/s)	100	99	96	90	80	50	0

Determine the following quantities: L_3, R_3, M_3

Then, use those quantities to determine both T_3 and S_3.

Sketch a plot of the points to aid your understanding. Clearly show the results of your computations in the space below (ideally, take a photograph of handwritten work to include here, as how you arrived at all 5 values is important to document). Also write one sentence to state which of the 5 estimates you think is the best possible estimate of how far the car traveled while braking.

(6) **Ans:** Based on the reasoning used in the previous problems I strongly believe that S_3 would be the most accurate result of the exact integral.



$\Delta x = \frac{1.8-0}{3} = 0.6$

$\int_0^{1.8} f(x)\,dx$

$Ln_3 = 0.6(100+96+80) = 165.6$

$T_3 = \frac{165.6+105.6}{2} = 135.6$

$R_3 = 0.6(96+80+0) = 105.6$

$S_3 = \frac{2(143.4+135.6)}{3} = 140.8$

$M_3 = 0.6(99+90+50) = 143.4$