Weekly Blog: Lebesgue's Criterion for Riemann Integrability

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Introduction

In this blog, I will look into Lebesgue's Criterion for Riemann Integrability, using the results we've established throughout this semester.

Previously, we have built the criterion for Riemann integrability based on the construction of sums and the equality of upper and lower integrals. But, recall Theorem 7.2.9, which states: if f is continuous on $[a, b]$, then it is integrable. Additionally, recall that the Dirichlet Function, which is non-integrable, is discontinuous at every point on $[0, 1]$. Then, perhaps there is a deeper connection between continuity (or discontinuities) and integrability.

Measure

The ultimate goal of this post is to construct a new criteria for the integral, specifically, exploring Lebesgue's criterion for integrability. But, before we can dive into Lebesgue's Theorem, we must recognize some new machinery.

Let's start with measure. The definition of a set with measure zero is: (Definition 7.6.1) A set $A \subseteq \mathbb{R}$ has measure zero if, for all $\epsilon > 0$, there exists a countable collection of open intervals O_n with the property that A is contained in the union of all of the intervals O_n and the sum of the length of all of the intervals is less than or equal to ϵ. More precisely, if $\mid O_n \mid$ refers to the length of the interval O_n, then we have

$$A \subseteq \bigcup_{n=1}^{\infty} O_n \text{ and } \sum_{n=1}^{\infty} \mid O_n \mid \leq \epsilon$$

α-continuity

In addition to measure, we are introduced with α-continuity: Let f be defined on $[a, b]$, and let $\alpha > 0$. The function f is α-continuous at $x \in [a, b]$ if there exists $\delta > 0$ such that for all $y, z \in (x-\delta, x+\delta)$ it follows that $\mid f(y)-f(z) \mid< \alpha$.

We define D^α to be the set of points in $[a, b]$ where the function f fails to be α-continuous, or

$$D^\alpha = \{x \in [a, b] : f \text{ is not } \alpha\text{-continuous at } x\}$$

Lebesgue's Theorem

Finally, we can arrive at Lebesgue's Theorem: Let f be a bounded function defined on the interval $[a, b]$. Then, f is Riemann-integrable if and only if the set of points where f is not continuous has a measure zero.

Let's look at how we could use the new definitions in this section, as well as older results that we've proven, to prove this theorem.

Proof Outline:

Let $M > 0$ satisfy $\mid f(x) \mid\leq M$ for all $x \in [a, b]$, and define $D^\alpha = \{x \in [a, b] : f \text{ is not } \alpha\text{-continuous at } x\}$ and $D = \{x \in [a, b] : f \text{ is not continuous at } x\}$.

$\Leftarrow$ Assume D has measure 0.
Let $\epsilon > 0$ be arbitrary. Set $\alpha = \frac{\epsilon}{2(b-a)}$.

We can apply the Heine-Borel Theorem to take a finite subcollection of disjoint open intervals $\{G_1, G_2, ..., G_N\}$ satisfying $\sum_{n=1}^{N} \mid G_n \mid< \frac{\epsilon}{4M}$.

Set $K = [a, b] \setminus \bigcup_{n=1}^{N} G_n$. Since K is closed and bounded, K is compact by the Heine-Borel Theorem. Similar to continuity, if a function is alpha-continuous on a compact set, then it is uniformly α-continuous on that compact set. Then, knowing f is α-continuous on K, f is uniformly α-continuous on K.

Finally, we can construct a partition P_ϵ of $[a, b]$ from the endpoints of the sub intervals covering D^α.

It follows that

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{\bigcup G_n} (M_k - m_k)\Delta x_k + \sum_{K} (M_k - m_k)\Delta x_k < \epsilon$$

so f is Riemann integrable by Integrability Criterion.

$\Rightarrow$ Suppose f is Riemann-integrable.

As usual, we let $\epsilon > 0$ be arbitrary, and we can fix $\alpha > 0$.

By the Integrability Criterion, $\exists P_\epsilon$ s.t. $U(f, P_\epsilon) - L(f, P_\epsilon) < \alpha\epsilon$.

Knowing that we want to prove a set is measure zero, we will have to construct sub intervals that satisfy our definition for sets of measure zero.

Instead of working with the set D, it will be easier to construct sub intervals containing points of D^α to show that D^α has measure 0.

Here we get to apply the Heine-Borel Theorem. Recall, that since $D^\alpha \subseteq [a, b]$ is closed (D^α is closed for $\alpha > 0$, a result from exercise 7.6.8) and bounded, it is possible to choose a cover for D^α that consists of a finite sub collection of open intervals.

Finally, once you have shown D^α has measure 0, we can show that D has measure 0.

Conclusive Results:

We have found a new method of characterizing integrablility, but how can we apply this to what we have already discovered? Well, let's reconsider the Dirichlet function on the interval $[0, 1]$. For all $n \in \mathbb{N}$, define

$$f_n(x) = \begin{cases} 1 \text{ if } x \in \{r_1, r_2, .., r_n\} \\ 0 \text{ otherwise} \end{cases}$$

Where $\{r_1, r_2, ..., r_n\}$ is the set rational points in this interval.

We have previously shown that Dirichlet's nowhere continuous function did not satisfy our conditions for integrability, so it's tempting to think that f_n also fails to be integrable. However, f_n only has a finite number of discontinuities. Then, its set of discontinuities has measure 0, so f_n is Riemann integrable by Lebesgue's Theorem.