Continuous Nowhere-Differentiable Functions

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1 Introduction

We have seen (Theorem 5.2.3) that if a function $g : A \to R$ is differentiable at a point $c \in A$, then g is continuous at c as well. However, the converse of this statement is not true. As a counter example, consider $f : R \to R$ with $f(x) := |x|$. It is easy to prove that f is not differentiable at 0.
What may come as more of a surprise is the fact that *most* continuous functions fail to be differentiable at any point. Thus the continuous functions we are accustomed to make up an insignificant fraction of the set of all continuous functions. (This parallels the fact that the more familiar rational numbers are vastly outnumbered by the irrationals on R.) The proof of this conjecture requires measure theory and other more advanced topics, so we do not discuss it here. Instead, we will discuss instances of continuous nowhere-differentiable functions and their connections to other areas of mathematics.

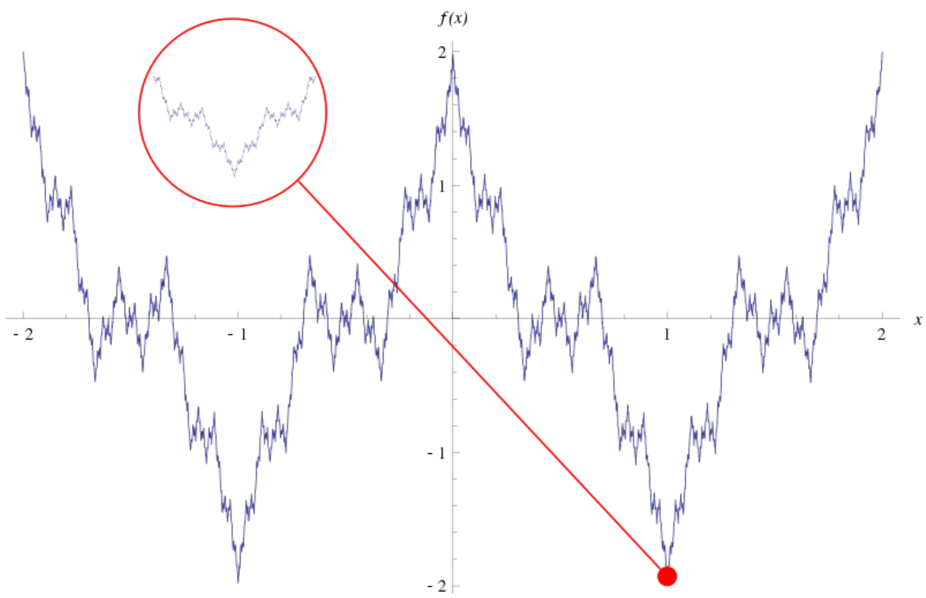
2 Two Examples

Our textbook provides a construction of a continuous nowhere-differentiable function. Define $h(x) := |x|$ on the interval $[-1, 1]$ and extend the domain of h to all of R by requiring that $h(x + 2) = h(x)$. Then the function

$$g(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

is continuous and nowhere-differentiable on R. The Algebraic Continuity Theorem can be invoked to show that the finite sum $\sum_{n=0}^{m} \frac{1}{2^n} h(2^n x)$ must be continuous on R for all $m \in N$. The book postpones to the next chapter a rigorous extension of this fact to the infinite case.
To prove lack of differentiability, we could first show that $g'(x)$ does not exist for any rational number of the form $x = \frac{p}{2^k}$, where $p \in Z$ and $k \in N \cup \{0\}$. We could then prove lack of differentiability for all other real numbers y by using the fact that for all $m \in N \cup \{0\}$, there exists some $p_m \in Z$ such that $\frac{p_m}{2^m} < y < \frac{p_m+1}{2^m}$. Combining this observation with a new sequential criterion

for lack of differentiability (Exercise 5.4.7) gives us the desired result.



Weierstrass function

A similar example of a continuous nowhere-differentiable function is

$$f(x) := \sum_{n=0}^{\infty} a^n \cos(b^n x)$$

provided $a \in (0,1)$, b is an odd integer, and $ab > 1+\frac{3\pi}{2}$. (This class of functions was first constructed by Karl Weierstrass in 1872.) Since $1+\frac{3\pi}{2} > 5.7$, we can see that the smallest possible value for b is 7. It may be more surprising that this second example turns out to be nowhere-differentiable. Our first function g was an infinite sum of functions who *themselves* were non-differentiable at an infinite number of points (the "corner points" of the repeated absolute value function). However f is the sum of functions that are differentiable on all of R. Hence any finite sum of these functions is continuous on all of R. (See the next section for plots of the first three partial sums for g and f.) This is just another instance of a property that holds true in every finite case not extending to the infinite case.

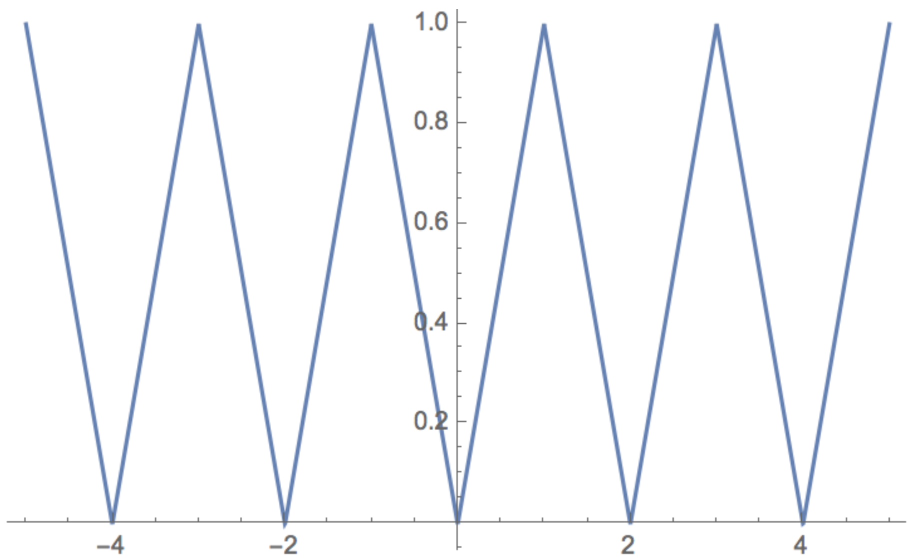
3 Plots of Partial Sums

1) Plots of

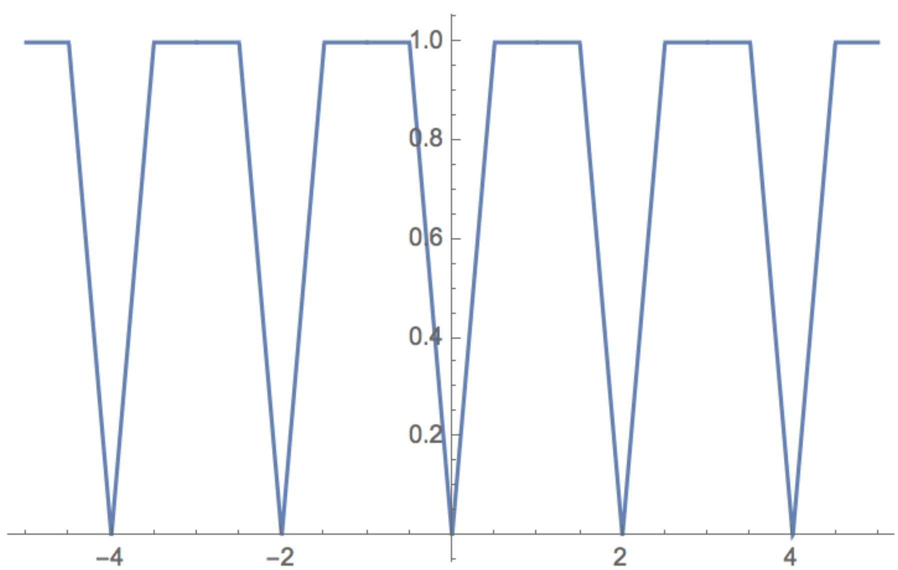
$$g_m(x) := \sum_{n=0}^{m} \frac{1}{2^n} h(2^n x)$$

for $m = 0, 1, 2$:

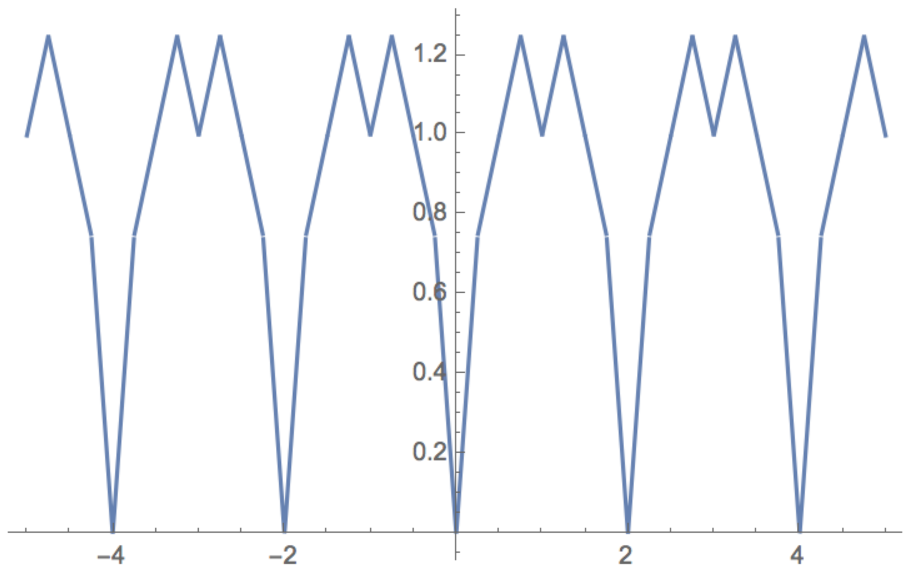
(Note that for each m, $g_m(x)$ has points of non-differentiability.)



(i) $m = 0$



(ii) $m = 1$



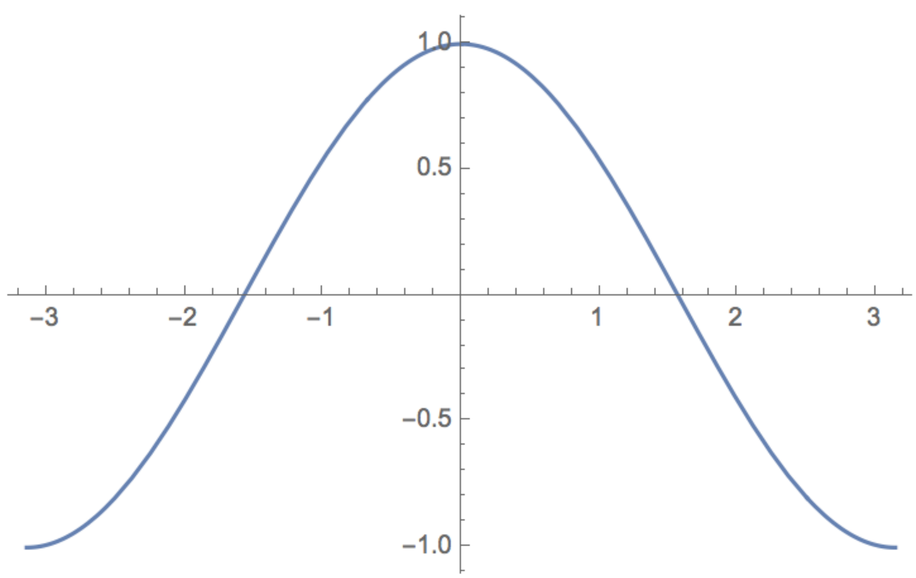
(iii) $m = 2$

2) Plots of

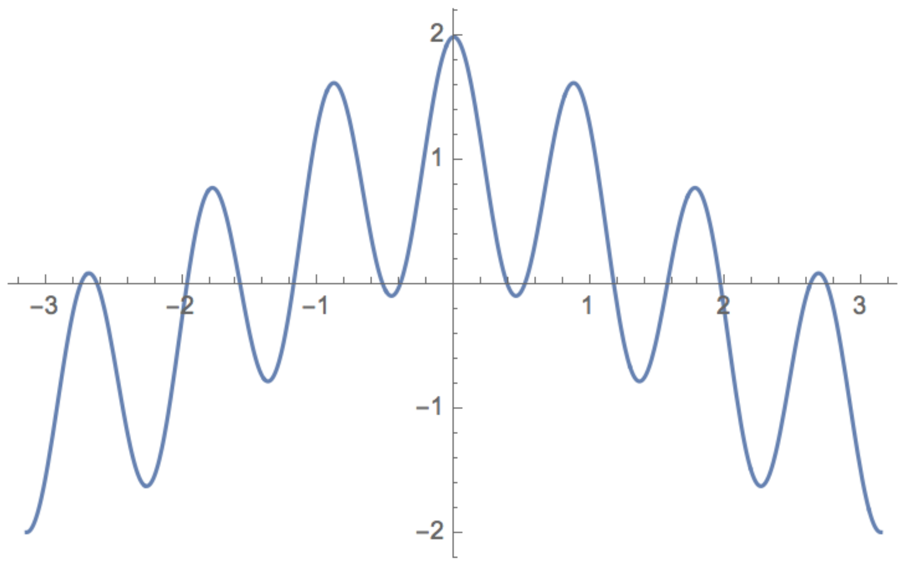
$$f_m(x) := \sum_{n=0}^{m} a^n \cos(b^n x)$$

for $m = 0, 1, 2$:

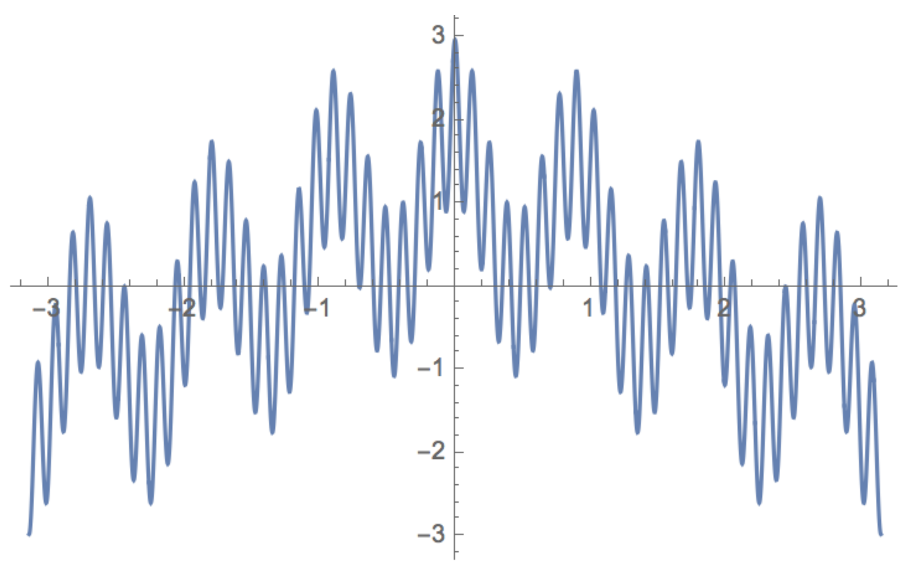
(Note that for each m, $f_m(x)$ is differentiable on all of R.)



(iv) $m = 0$



(v) $m = 1$

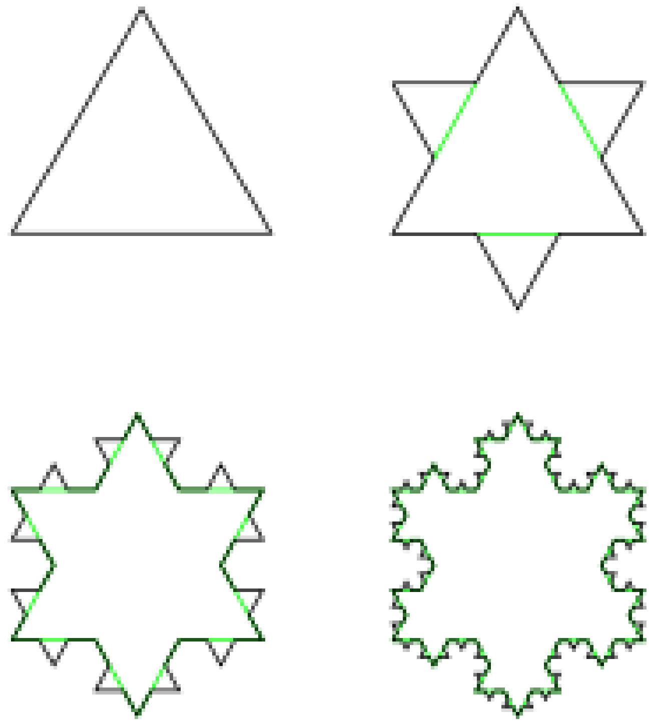


(vi) $m = 2$

4 Fractals and Brownian Motion

It might come as a surprise but examples of continuous nowhere-differentiable functions exist in nature. One of the most prominent example is of the Brownian motion. Generally, Brownian motion is simply described as the random motion of particles suspended in a fluid resulting from their collision with the atoms or molecules in the fluid. However, the Wiener process provides a mathematical definition of the Brownian motion as a stochastic process. According to a paper written by Aaron McKnight (http://www.math.uchicago.edu/ may/VIGRE/VIGRE2009/REUPapers/McKnight.pdf), it is not too difficult to produce a stochastic process that fulfills the properties of the Wiener process, but it is nontrivial to show that it is everywhere continuous. Quite astonishingly, McKnight manages to prove in his paper that if the Brownian motion is defined by the Wiener process then it is not only everywhere continuous, but the randomness allows it to also be nowhere differentiable.

Another example of a continuous nowhere-differentiable function is the Koch curve. This curve is obtained from a line segment by repeating the following three steps indefinitely (taken from https://en.wikipedia.org/wiki/Koch_snowflake):
(i) Divide the line segment into three segments of equal length.
(ii) Draw an equilateral triangle that has the middle segment from step 1 as its base and points outward.
(iii) Remove the line segment that is the base of the triangle from step 2.
At first glance, you may notice that this curve is not in fact a function. That is, if you draw the Koch curve in the xy-plane, you will notice that some values of x will correspond to multiple values of y. Thus when we say the Koch curve is continuous and nowhere-differentiable, we refer to the *parameterized* Koch curve.



Koch's snowflake

Now consider the Koch snowflake, obtained from repeating ad infinitum the three steps above on each of the three sides of an equilateral triangle. The image above shows the first three repetitions of these steps. For example, the equilateral triangle on the top left of the figure is transformed by dividing each side into three segments and drawing outward pointing equilateral triangles from the middle segment of each side. It turns out that the Koch snowflake has a finite area but an infinite perimeter! To be exact, the area of the figure converges to $\frac{8}{5}A_0$, where A_0 is the area of the initial equilateral triangle (http://mathworld.wolfram.com/KochSnowflake.html). This is a byproduct of the Koch curve's non-differentiability–any "well-behaved" closed curve would certainly be expected to have a finite perimeter. Note that if we consider a curve that is not a closed curve, we can make similar constructions that are differentiable at most points. For example, take the region bounded by the x and y axes, and the curve $y = e^{-x}$. The area enclosed by these curves is finite, while the perimeter is infinite. Further, the parameterized curve fails to be differentiable only at the points $(0, 0)$ and $(0, 1)$.

5 References

1) http://www.math.uchicago.edu/ may/VIGRE/VIGRE2009/REUPapers/McKnight.pdf
2) http://mathworld.wolfram.com/KochSnowflake.html
3) https://en.wikipedia.org/wiki/Koch_snowflake
4) https://en.wikipedia.org/wiki/Weierstrass_function
5) Stephen Abbott, *Understanding Analysis*