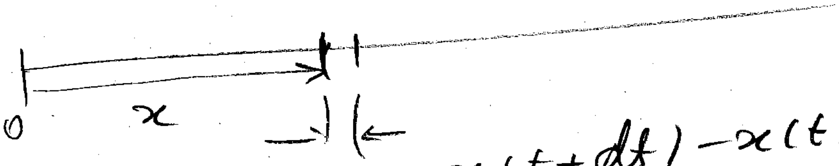
Lecture 3

The Speed of the Brownian particle

In a $\boxed{\text{classical}}$ motion with a constant speed

$$x = vt$$

so



$$dx = x(t+dt) - x(t)$$

$$dx = v\,dt$$

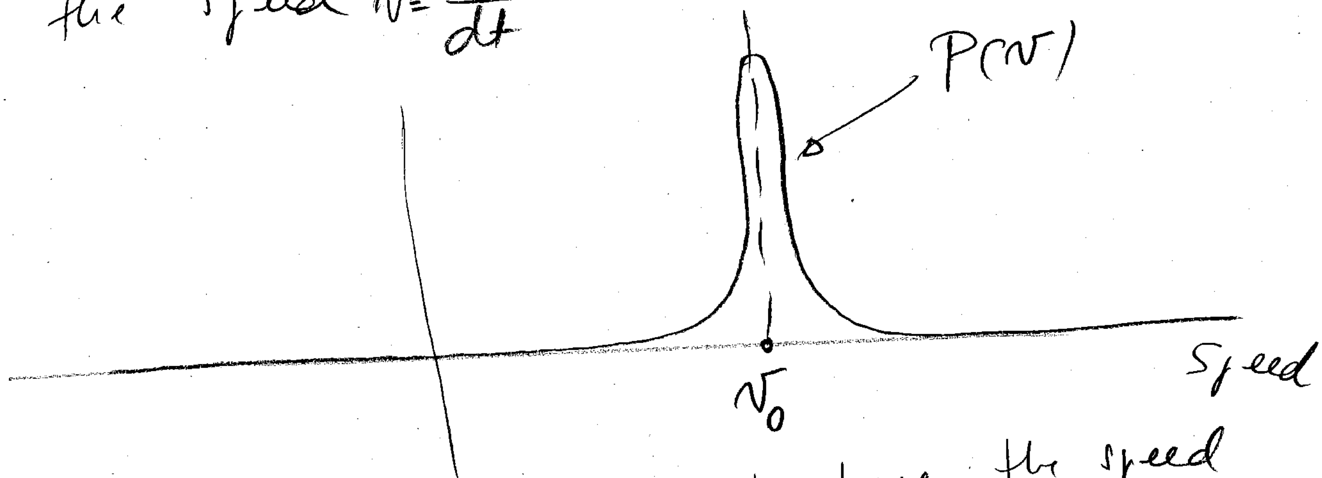
In other words, if the particle was at x at time t it will be for sure at $x+dx$ at time $t+dt$. The probability to have a displacement $dx = v\,dt$ in the time interval dt is 1.

In a $\boxed{\text{Brownian}}$ motion if we know dt (lets say $dt = 0.01$ seconds) then it does not follow a precise displacement dx. This dx can take any value from $-\infty$ to $+\infty$, with some probability.

The speed $\frac{dx}{dt}$ can thus take any value from $-\infty$ to $+\infty$.

Think about two extreme cases for the probability density for

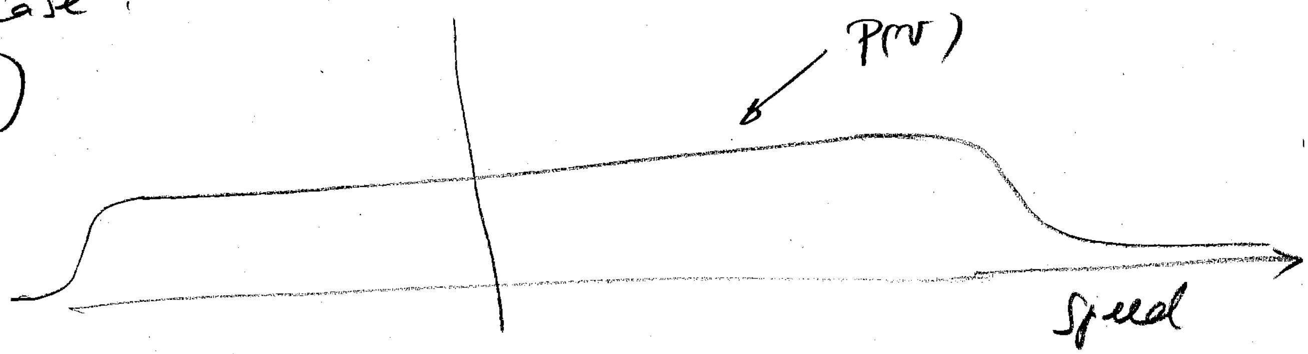
ⓐ The probability density for the speed $v = \frac{dx}{dt}$ is like



That is, the probability to have the speed v_0 is high, and other values for away from v_0 occur not very often.

This case is close to the deterministic case. We almost know the speed.

ⓑ



Here many values of the speed have the same chance to appear.

The speed is

$$v = \frac{dx}{dt}$$

and because dx (the increment) is a Gaussian with mean $=0$ and variance $= dt$, the speed v will be also a Gaussian.

Remark

If a random variable x is Gaussian

with $$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

Now put $y = \frac{x}{b}$ with b some fixed number (b is NOT random). Then y is Gaussian also by a simple change of variables.

$$\text{Mean } y = \frac{\text{Mean } x}{b}$$

$$\text{Variance } y = \frac{\text{Variance } x}{b^2}$$

So

$$p(y) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{b^2}}} e^{-\frac{x^2}{2\frac{\sigma^2}{b^2}}}$$

Back to the speed, dt is like the fixed number b and we know that $\sigma^2 = 2D\,dt$ for dx.

Then

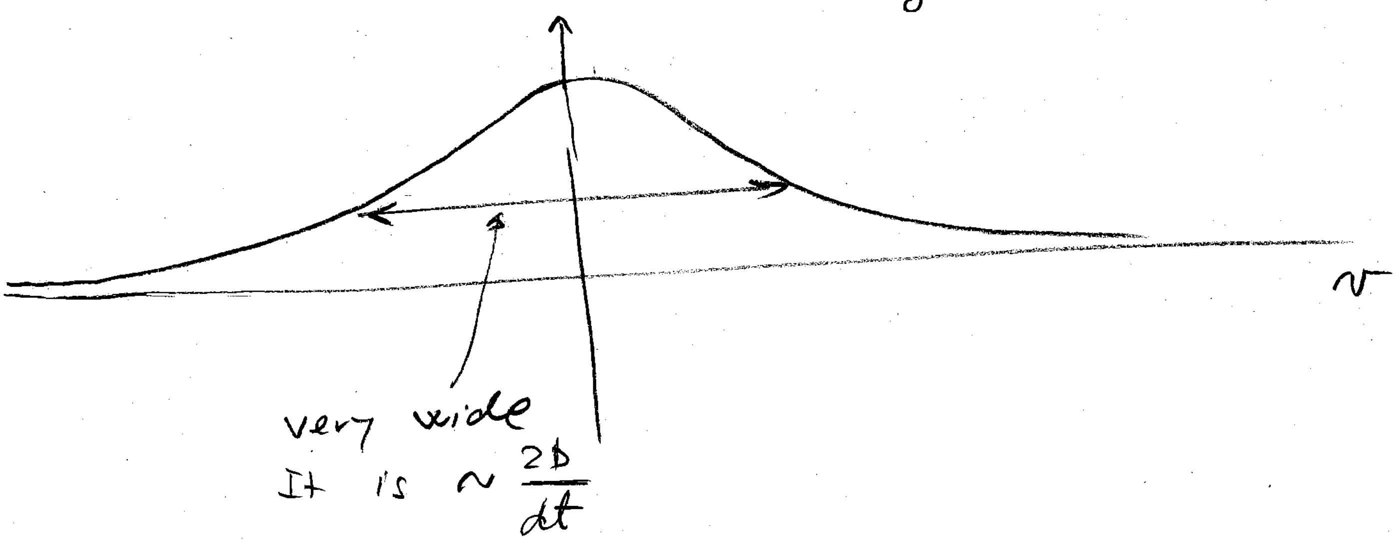
$$p(v) = \frac{1}{\sqrt{2\pi \frac{2D\,dt}{dt^2}}} e^{-\frac{v^2}{2\frac{2D\,dt}{dt^2}}}$$

So the probability density for the speed in the time interval dt is

$$\boxed{P(v) = \frac{1}{\sqrt{2\pi \frac{2D}{dt}}} e^{-\frac{v^2}{2 \cdot \frac{2D}{dt}}}}$$

$$\sigma^2_{\text{for the speed}} = \frac{2D}{dt}$$

Conclusion: if $dt \to 0$ the speed dispersion $\to \infty$!



The Wiener Process

$$\frac{\partial}{\partial t} P(w,t|w_0,t_0) = \frac{1}{2}\frac{\partial^2}{\partial w^2} P(w,t|w_0,t_0)$$

Compare this equation for the Wiener process with the Einstein equation

$$\frac{\partial}{\partial t} P(x,t|0,0) = D \frac{\partial^2}{\partial x^2} P(x,t|0,0)$$

and its solution

$$P(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}$$

We get

$$D = \frac{1}{2}$$ for the Wiener process

why need $D = \frac{1}{2}$?

Because then

$$P(x,t|0,0) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

which is a Gaussian probability density with

$$\sigma^2 = t$$

Noise

A random variable

Noise = (Strength) $\cdot$ (Speed of Brownian Motion)

a number measuring the strength of the noise

$$\sigma^2_{Noise} = \langle (Noise)^2 \rangle = (Strength)^2 \langle v^2 \rangle$$
$$= (Strength)^2 \cdot \frac{1}{dt}$$

$$\mu_{Noise} = \langle Noise \rangle = 0$$

So

$$P(Noise \mid dt) = \frac{1}{\sqrt{2\pi \frac{(Strength)^2}{dt}}} \, e^{-\frac{Noise}{2\pi \frac{(Strength)^2}{dt}}}$$

probability density to have an fluctuation increment, in a small time interval

$$\sigma = \frac{Strength}{\sqrt{dt}}$$

dt

Solutions of the Langevin Equation by Computer Simulation

$$\frac{dv}{dt} = -\gamma v + \text{Strength} \cdot \xi(t)$$

the speed of a Wiener process

speed here also

speed here

So

$$dv = -\gamma v\,dt + \text{Strength}\ \xi(t) \cdot dt$$

You can generate samples for $\xi(t)$ or for $\xi(t) \cdot dt = dW$. The speed and the increments have different probability densities

$$\langle (\xi(t))^2 \rangle = \frac{1}{dt}$$

$$\langle (dW)^2 \rangle = \langle (\xi(t))^2 (dt)^2 \rangle = dt$$

the mean of both is zero

So

$$P(\xi) = \frac{1}{\sqrt{2\pi \frac{1}{dt}}}\, e^{-\frac{\xi^2}{2\frac{1}{dt}}}$$

$$P(dW) = \frac{1}{\sqrt{2\pi\, dt}}\, e^{-\frac{\xi^2}{2\,dt}}$$