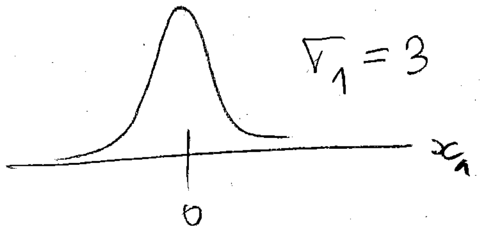
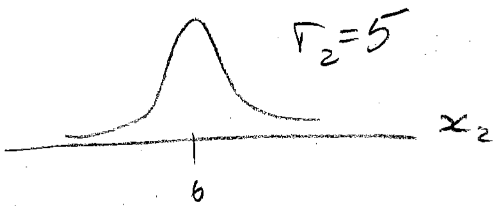
Gaussian Probability density



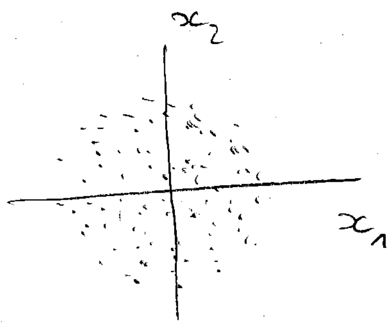
$$P(x_1) = \frac{1}{\sqrt{2\pi\,\sigma_1^2}}\, e^{-\frac{x_1^2}{2\sigma_1^2}}$$

$\langle x_1 \rangle = 0$; $\langle (x_1 - \langle x_1 \rangle)^2 \rangle = \sigma_1^2$



$$P(x_2) = \frac{1}{\sqrt{2\pi\,\sigma_2^2}}\, e^{-\frac{x_2^2}{2\sigma_2^2}}$$

$\langle x_2 \rangle = 0$; $\langle (x_2 - \langle x_2 \rangle)^2 \rangle = \sigma_2^2$



$$P(x_1, x_2) = P(x_1)\, P(x_2)$$

$$\langle x_1 x_2 \rangle = \langle x_1 \rangle \langle x_2 \rangle = 0 \cdot 0 = 0$$

or $\langle (x_1 - \langle x_1 \rangle)(x_2 - \langle x_2 \rangle) \rangle = 0$

<u>No correlation</u>

$$P(x_1, x_2) = \frac{1}{\sqrt{(2\pi)^2 \sigma_1^2 \sigma_2^2}}\, e^{-\left[\frac{x_1^2}{2\sigma_1^2} + \frac{x_2^2}{2\sigma_2^2}\right]} =$$

$$= \text{Constant} \quad \exp\left[-\frac{1}{2} \left(x_1 (\sigma_1^2)^{-1} x_1 + x_2 (\sigma_2^2)^{-1} x_2 \right) \right]$$

$$\text{Correlation matrix} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} =$$

$$= \begin{bmatrix} \langle (x_1 - \langle x_1 \rangle)(x_1 - \langle x_1 \rangle) \rangle & \langle (x_1 - \langle x_1 \rangle)(x_2 - \langle x_2 \rangle) \rangle \\ \langle (x_2 - \langle x_2 \rangle)(x_1 - \langle x_1 \rangle) \rangle & \langle (x_2 - \langle x_2 \rangle)(x_2 - \langle x_2 \rangle) \rangle \end{bmatrix}$$

$$\text{Correlation matrix} = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 25 \end{bmatrix}$$

$$\varphi_{11} = \sigma_1^2 = 9, \quad \varphi_{12} = 0, \quad \varphi_{21} = 0, \quad \varphi_{22} = \sigma_2^2 = 25$$

$$P(x_1, x_2) = \underbrace{\frac{1}{\sqrt{(2\pi)^2\, 9 \cdot 25}}}_{\text{Constant}} \exp\left[-\frac{1}{2}\left(x_1 (9)^{-1} x_1 + x_2 (25)^{-1} x_2\right)\right]$$

$$P(x_1, x_2) = \text{Constant} \exp\left[-\frac{1}{2}\left(x_1 (\varphi^{-1})_{11} x_1 + x_2 (\varphi^{-1})_{22} x_2\right)\right]$$

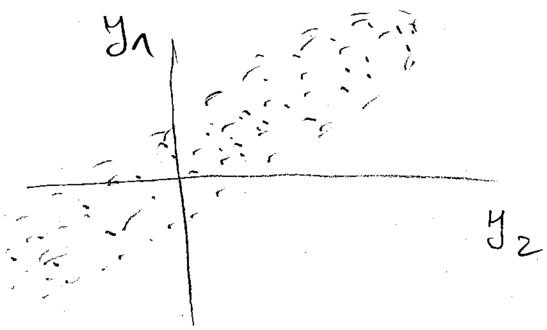
$$\varphi^{-1} = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} \Rightarrow \begin{aligned} (\varphi^{-1})_{11} &= \frac{1}{9} \\ (\varphi^{-1})_{22} &= \frac{1}{25} \end{aligned}$$

Include correlations between random variables

$$y_1 = a x_1 + b x_2$$
$$y_2 = c x_1 + d x_2$$

Example

$$(1) \quad \begin{cases} y_1 = 2x_1 + x_2 \\ y_2 = -x_1 + x_2 \end{cases}$$



$$\langle y_1 \rangle = 2\langle x_1 \rangle + \langle x_2 \rangle = 0$$
$$\langle y_2 \rangle = -\langle x_1 \rangle + \langle x_2 \rangle = 0$$
$$\langle y_1 y_2 \rangle = \langle -2x_1^2 + 2x_1x_2 - x_1x_2 + x_2^2 \rangle = -2 \cdot 9 + 25 = 7$$

Now solve for x_1 and x_2

$$y_1 - y_2 = 3x_1 \quad \Rightarrow \quad \boxed{x_1 = \frac{1}{3}(y_1 - y_2)}$$

$$x_2 = y_2 + x_1 = y_2 + \frac{1}{3}y_1 - \frac{1}{3}y_2$$

$$\boxed{x_2 = \frac{1}{3}y_1 + \frac{2}{3}y_2}$$

So

$$P(y_1, y_2) = \text{Constant} \cdot \exp\left[-\frac{1}{2} \left(x_1 (9^{-1}) x_1 + x_2 (25)^{-1} x_2 \right) \right] =$$

$$= \text{Const} \cdot \exp\left[-\frac{1}{2} \left(\left(\frac{1}{3} y_1 - \frac{1}{3} y_2\right) (9^{-1}) \left(\frac{1}{3} y_1 - \frac{1}{3} y_2\right) + \right.\right.$$

$$\left.\left. + \left(\frac{1}{3} y_1 + \frac{2}{3} y_2\right) (25^{-1}) \left(\frac{1}{3} y_1 + \frac{2}{3} y_2\right) \right) \right] =$$

$$= \text{Const} \cdot \exp\left[-\frac{1}{2} \left(y_1 \left(\frac{1}{3} (9^{-1}) \frac{1}{3} + \frac{1}{3} (25)^{-1} \frac{1}{3} \right) y_1 + \right.\right.$$

$$+ y_1 y_2 \left(-\frac{1}{3} (9^{-1}) \frac{1}{3} - \frac{1}{3} (9^{-1}) \frac{1}{3} + \frac{1}{3} (25)^{-1} \frac{2}{3} + \frac{1}{3} (25)^{-1} \frac{2}{3} \right)$$

$$\left.\left. + y_2 \left(\frac{1}{3} (9^{-1}) \frac{1}{3} + \frac{2}{3} (25)^{-1} \frac{2}{3} \right) \right) \right] \qquad (\text{Expression } *)$$

Now find the correlation matrix for y_1, y_2

$$\theta = \begin{bmatrix} \langle (y_1 - \langle y_1 \rangle)(y_1 - \langle y_1 \rangle) \rangle & \langle (y_1 - \langle y_1 \rangle)(y_2 - \langle y_2 \rangle) \rangle \\ \langle (y_2 - \langle y_2 \rangle)(y_1 - \langle y_1 \rangle) \rangle & \langle (y_2 - \langle y_2 \rangle)(y_2 - \langle y_2 \rangle) \rangle \end{bmatrix}$$

$$\theta_{11} = \langle (y_1 - \langle y_1 \rangle)^2 \rangle = \langle (y_1)^2 \rangle = \langle (2x_1 + x_2)^2 \rangle = 4 \langle x_1^2 \rangle + \langle x_2^2 \rangle =$$

$$= 4 \cdot 9 + 25 = 36 + 25 = 61$$

$$\theta_{12} = \langle y_1 y_2 \rangle = -2 \langle x_1^2 \rangle + \langle x_2^2 \rangle = -2 \cdot 9 + 25 = 7$$

$$\theta_{22} = \langle (y_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle = 9 + 25 = 34$$

$$\theta = \begin{bmatrix} 61 & 7 \\ 7 & 34 \end{bmatrix}$$

$$\theta^{-1} = \begin{bmatrix} \frac{34}{2025} & -\frac{7}{2025} \\ -\frac{7}{2025} & \frac{61}{2025} \end{bmatrix}$$

Now write $P(y_1, y_2)$ in terms of

$$(\theta^{-1})_{11} = \frac{34}{2025}, \quad (\theta^{-1})_{12} = -\frac{7}{2025}$$

$$(\theta^{-1})_{21} = -\frac{7}{2025}, \quad (\theta^{-1})_{22} = \frac{41}{2025}$$

Use Expression $*$:

$$\boxed{P(y_1, y_2) = \text{Constant} \cdot \exp\left[-\frac{1}{2}\left(y_1 \cdot (\theta^{-1})_{11}\, y_1 + y_1 (\theta^{-1})_{12}\, y_2 + y_2 (\theta^{-1})_{21}\, y_1 + y_2 (\theta^{-1})_{22}\, y_2\right)\right]}$$

You can now work an example with nonzero mean

$$y_1 = 2x_1 + x_2 + 1$$

$$y_2 = -x_1 + x_2 - 3$$

So

$$\langle y_1 \rangle = 1$$

$$\langle y_2 \rangle = -3$$

Denote $\mu_1 = 1$ (the mean of Y_1)

$\mu_2 = -3$ (the mean of Y_2)

and

$$\gamma = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

the correlation matrix that you need to compute

You will find for this example

$$P(y_1, y_2) = \text{Constant} \cdot \exp\Big[-\frac{1}{2}\Big((y_1-\mu_1)\,(\gamma^{-1})_{11}\,(y_1-\mu_1) +$$

$$+ (y_1-\mu_1)\,(\gamma^{-1})_{12}\,(y_2-\mu_2) + (y_2-\mu_2)\,(\gamma^{-1})_{21}\,(y_1-\mu_1) +$$

$$+ (y_2-\mu_2)\,(\gamma^{-1})_{22}\,(y_2-\mu_2)\Big)\Big]$$