Gaussian Noise

From the discussions about "noise" you realized that to be able to compute the dispersion of a physical parameter (like the speed in Langevin's equation) you need to specify the correlation functions for the noise:

$$\langle X(t_1)\, X(t_2) \cdots X(t_n) \rangle \; . \qquad (1)$$

In other words you need to find either from experiment or from some theoretical models the multitime correlation functions (1).

Unfortunatelly the list of all corellation functions grows rapidly with the number of time points $t_1, \ldots, t_n$.

Fortunatelly for a class of noise that is often encounter in practice the list is short. You only need to know the two-point correlation function; all the other multitime correlation functions can be easily computed from the two-point correlation functions. This class of noise is called Gaussian noise.

On page 22 from the book of R. Kubo, M. Toda and N. Hashitsume, "Statistical Physics II, Nonequilibrium Statistical Mechanics" you find a simple and at the point explanation of the Gaussian noise (called in the book Gaussian Processes). Please consult your hand outs.

In what follow I will explain the main points by an example.

<u>First</u>. The stochastic process $Z(t)$ is Gaussian if the probability distribution at n of its observed values $z_1, z_2, \ldots, z_n$ time points $t_1, t_2, \ldots, t_n$ is an n-dimensional Gaussian distribution:

$$W_n(z_1, t_1; z_2, t_2; \ldots; z_n, t_n) =$$

$$= \text{Constant} \cdot \exp\left[-\frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} (z_j - \mu_j)(\varphi^{-1})_{jk} (z_k - \mu_k)\right.$$

where μ_j depends on the time t_j and is the average value of the stochastic process (noise here) at time t_j:

$$\mu_j(t_j) = \langle z_j(t_j) \rangle$$

The 2-time correlation functions of the stochastic process (noise) are

$$\varphi_{jk}(t_j, t_k) = \langle [z(t_j) - \langle z(t_j)\rangle][z(t_k) - \langle z(t_k)\rangle] \rangle$$

Now we make the <u>hypothesis</u> that our noise <u>is</u> <u>Gaussian</u> and that we know the 2-time correlation functions

$$\varphi_{jk}(t_j, t_k).$$

[<u>Remark</u>: in Kubo's book the matrix φ is denoted as A^{-1}]

<u>Goal</u>: Find the <u>multipoint</u> correlation functions in terms of the <u>known</u> 2-point correlation functions $\varphi_{jk}(t_j, t_k)$.

<u>Procedure</u>

(I) To find the multipoint correlation functions $\langle z(t_1) z(t_2) \cdots z(t_n)\rangle$ we will use the generating function

$$(1) \quad \Phi(J_1, J_2, \ldots, J_n) = \int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_n \, W_n(z_1, t_1; \ldots; z_n, t_n) \cdot \exp\left(i \sum_{j=1}^{n} J_j z_j\right)$$

Do the integral (1) and you will find (see the hand out)

$$(2)\ \Phi(J_1, J_2, \dots, J_n) = \exp\left(i \sum m_j(t_j) J_j - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \varphi(t_j, t_k) J_j J_k\right)$$

(II) How do we use (1) and (2) from above?

In (1), expand $\exp\left(i \sum_{j=1}^{n} J_j z_j\right)$ in a power series

$$\exp\left(i \sum_{j=1}^{n} J_j z_j\right) = \sum_{r_1=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \frac{(iJ_1)^{r_1} \cdots (iJ_n)^{r_n}}{r_1! \cdots r_n!} z_1^{r_1} \cdots z_n^{r_n}$$

and do the integral in (1)

$$(3)\ \Phi(J_1, J_2, \dots, J_n) = \int_{-\infty}^{\infty} dz\, W \sum_{r} \frac{(iJ_1)^{r_1} \cdots (iJ_n)^{r_n}}{r_1! \cdots r_n!} z_1^{r_1} \cdots z_n^{r_n} =$$

$$= \sum_{r} \frac{(iJ_1)^{r_1} \cdots (iJ_n)^{r_n}}{r_1! \cdots r_n!} \langle z_1^{r_1} \cdots z_n^{r_n} \rangle$$

This is in general, for every probability W.

Now, if you know ϕ (AS IN (2)), then you will expand it in powers of $J_1, \dots, J_n$, and read the coefficients to obtain $\langle z_1^{r_1} \cdots z_n^{r_n} \rangle$.

Now expand it using

$$e^{y} = 1 + \frac{1}{1!} y + \frac{1}{2!} y^2 + \frac{1}{3!} y^3 + \cdots$$

with $y = [\cdots]$.

So

$$(5)\ \phi(J_1, J_2, J_3, J_4) = 1 + \frac{1}{1!}[\cdots] + \frac{1}{2!}[\cdots]^2 + \frac{1}{3!}[\cdots]^3 + \cdots$$

We are set to find the 4-time correlation function.

$$\langle z_1 z_2 z_3 z_4]$$

This will be the coefficient of (see (3))

$$\frac{(iJ_1)(iJ_2)(iJ_3)(iJ_4)}{1!\quad 1!\quad 1!\quad 1!}$$

which is $J_1 J_2 J_3 J_4$, in (5).

In (5) the product $J_1 J_2 J_3 J_4$ appears <u>ONLY</u> in the term $\frac{1}{2!}[\cdots]^2$. From (4) we find all the possibilities for $J_1 J_2 J_3 J_4$ to appear in $\frac{1}{2!}[\cdots]^2$. There are three possibilities

$$\begin{aligned} J_1 J_2 J_3 J_4 = &(J_1 J_2)\cdot(J_3 J_4) \\ &(J_1 J_3)\ (J_2 J_4) \\ &(J_1 J_4)\ (J_2 J_3) \end{aligned}$$

Conclusion: the 4-time correlation function is given in terms of the 2-time correlation function as

$$\langle z(t_1) z(t_2) z(t_3) z(t_4) \rangle =$$

$$= \varphi(t_1, t_2)\, \varphi(t_3, t_4) + \varphi(t_1, t_3)\, \varphi(t_2, t_4) + \varphi(t_1, t_4)\, \varphi(t_2, t_3).$$

In general

$$\langle z(t_1) \cdots z(t_n) = \begin{cases} 0 & \text{for odd } n \\ \sum\limits'_{\text{pairings}} \prod\limits_{\text{pairs}} \phi(t_j, t_k) & \text{for even } n \end{cases}$$

III. Example for 4-time correlation function

So we have t_1, t_2, t_3 and t_4.
To go straightforward to the point, we will take

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0,$$

That is we deal with a noise with zero mean, which is quite useful.

So

$$(4)\quad \Phi(J_1, J_2, J_3, J_4) = \exp\Big[-\tfrac{1}{2}\varphi(t_1,t_1)J_1J_1 - \tfrac{1}{2}\varphi(t_1,t_2)J_1J_2 - \underbrace{\tfrac{1}{2}\varphi(t_2,t_1)}_{\text{this is equal with } \varphi(t_1,t_2) \text{ because } \langle z_1 z_2\rangle = \langle z_2 z_1\rangle} J_2J_1 - \dots\Big]$$

$$= \exp\Big[-\tfrac{1}{2}\varphi(t_1,t_1)J_1J_1 - \varphi(t_1,t_2)J_1J_2 - \varphi(t_1,t_3)J_1J_3 - \varphi(t_1,t_4)J_1J_4 - \tfrac{1}{2}\varphi(t_2,t_2)J_2J_2 - \varphi(t_2,t_3)J_2J_3 - \varphi(t_2,t_4)J_2J_4 - \varphi(t_3,t_3)J_3J_3 - \varphi(t_3,t_4)J_3J_4 - \tfrac{1}{2}\varphi(t_4,t_4)J_4J_4\Big] = \exp\underbrace{[\dots]}_{\text{just a notation}}$$

for all the φ's in the bracket