Stochastic Processes

A general stochastic process:

You need to give the probabilities for all possible sets of

$$t_1, t_2, \ldots, t_n \qquad n = 1, 2, \ldots$$

$$W_1(x, t_1)\, dx = \Pr[x < x(t_1) \leq x + dx]$$

$$W_2(x_1, t_1; x_2, t_2)\, dx_1\, dx_2 =$$

$$= \Pr[x_1 < x(t_1) \leq x_1 + dx_1 \text{ AND } x_2 < x(t_2) \leq x_2 + dx_2]$$

$$\vdots$$

$$W_n(x_1, t_1; x_2, t_2; \ldots, x_n, t_n)\, dx_1\, dx_2 \ldots dx_n =$$

$$= \Pr[x_1 < x(t_1) \leq x_1 + dx_1, \text{ AND } \ldots \; x_n < x(t_n) < x_n + dx_n]$$

This is called the joint probability density for n random variables. $x(t_1), \ldots, x(t_n)$.

Brownian motion

The probability of finding a particle at x_n at time t_n when it was certainly located at x_{n-1} at t_{n-1} is independent of the knowledge of where the particle was before t_{n-1} (that is at $t_{n-2}, t_{n-3}, \ldots$)

So

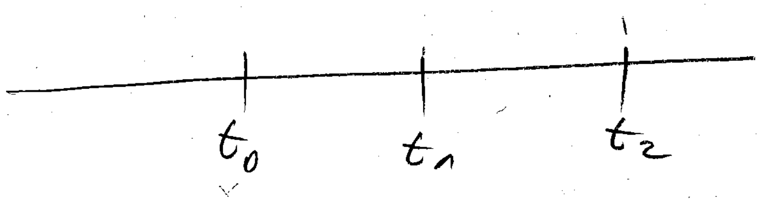
$$P(x_n, t_n \mid x_{n-1}, t_{n-1}; x_{n-2}, t_{n-2}; \ldots; x_0, t_0) =$$

$$= P(x_n, t_n \mid x_{n-1}, t_{n-1})$$

In othe words you need to know only the 2- time point transition probability to generate the entire process. In practice you find this transition probability from experiment or you give it to yourself when you construct a theoretical model

Once you know the 2-time point transition probability you have, for example

$$(1) \quad P(x_0, t_0 \mid x_1, t_1; x_2, t_2) = P(x_0, t_0 \mid x_1, t_1) \cdot P(x_1, t_1 \mid x_2, t_2)$$



IMPORTANT

Transition probability for 2-time points

$$P(x_1, t_1 \mid x_0, t_0)\, dx_1 = \frac{W_2(x_0, t_0; x_1, t_1)\, dx_1}{W_1(x_0, t_0)}$$

Example

$$P(x_1, t_1 \mid x_0, t_0) = \frac{1}{\sqrt{2\pi \cdot 2D(t-t_0)}}\, e^{-\frac{(x-x_0)^2}{2 \cdot 2D(t-t_0)}}$$

A general definiton of transition probabilities (n-time points)

$$P(x_1, t_1; x_2, t_2; \dots, x_n, t_n \mid x_0, t_0) =$$

$$= \frac{W_{n+1}(x_0, t_0; x_1, t_1; \dots, x_n, t_n)\, dx_1\, dx_2 \dots dx_n}{W_1(x_0, t_0)}$$

The initial state is precisely defined at time t_0 and we make n observations at n time points.

Conditional probability to be at x_n, t_n when a whole history lies behind

$$P(x_n, t_n \mid x_{n-1}, t_{n-1}; x_{n-2}, t_{n-2}; \dots; x_0, t_0)$$

The evolution of the process in the time interval (t_0, t_2) can be constructed be evolution in the two intervals (t_0, t_1) and (t_1, t_2), where t_1 is an arbitrary time point between t_0 and t_2.

therefore, integrating over all possible values of x_1 (the random variable) at t_1 gives

$$(2) \quad P(x_2, t_2 \mid x_0, t_0) = \int P(x_2, t_2 \mid x_1, t_1)\, dx_1\, P(x_1, t_1 \mid x_0, t_0)$$

A stochastic process is called Markovian if it satisfies conditions (1) (2).