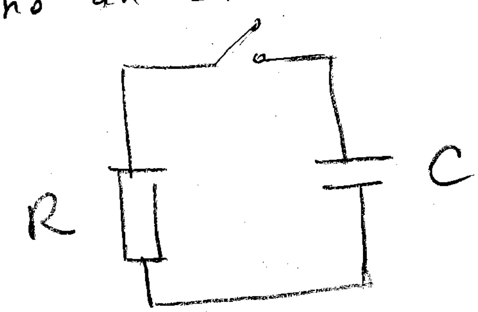
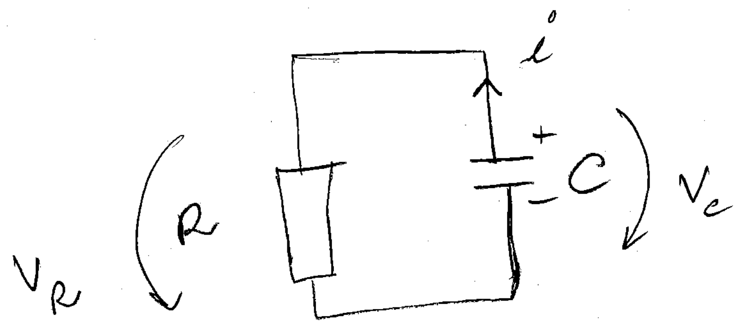
Noise in electrical circuits

In 1928 J.B Johnson published a detailed paper on the statistical fluctuation of electric charge in resistors (called by Johnson "conductors"). His paper describes detailed experimental procedures as well as the experimental formula for the noise intensity. The next paper in the same journal (Physical Review, volume 32, page 110) is by H. Nyquist. Nyquist gives a very elegant theoretical explanation of the Johnson's experimental results. In what follows I will describe the results from the present point of view.

Consider the following circuit. Notice that there is no an electromotive force acting on it (no battery).



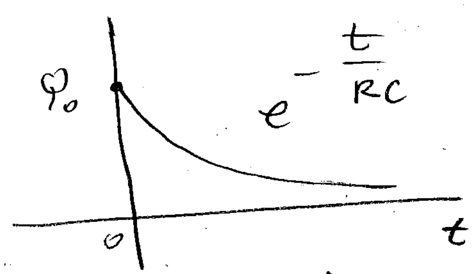
Suppose that the Capacitor is charged with the charge Q_0 at time $t=0_-$ (zero minus, just before zero). At $t=0$ the switch is closed and the charge will flow from one plate to another, through the resistor R. A current will flow through the circuit.



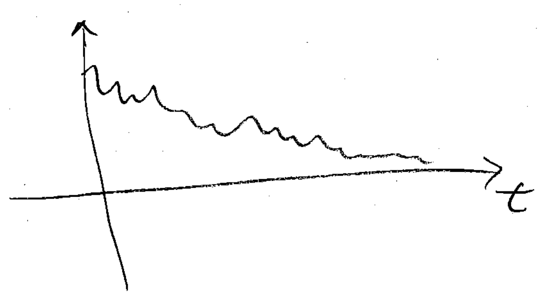
$$V_R = iR$$
$$V_C = \frac{\varphi}{C}$$
$$i = -\frac{d\varphi}{dt}$$

So
$$\frac{d\varphi}{dt} = -\frac{\varphi}{RC} \qquad (1)$$

with the solution
$$\varphi = \varphi_0 \, e^{-\frac{t}{RC}}$$



From Johnson's experiments we learn that the current i is noisy



To account for the experimental observation, we will add a Langevin noise $L(t)$ to (1)

$$\frac{dq}{dt} = -\frac{q}{RC} + L(t)$$

with

$$\langle L(t) \rangle = 0$$

$$\langle L(t) L(t') \rangle = \Gamma \, \delta(t-t')$$

The experimental measurements show that

$$\boxed{\Gamma = \frac{2kT}{R}} \qquad (*)$$

Here k = Boltzmann's constant
T = the absolute temperature of the resistor
R = the value of the resistor

The fact that the noise $(*)$ depends only on the parameter $\underline{\underline{R}}$ of the resistor alone, has led to the picture of a "noise source" located in the resistor. This "noise source" may be regarded as a current source producing a fluctuating current $I(t)$ to be added to the macroscopic current.

That is (1) is written as

$$\underbrace{\frac{d\varphi}{dt}}_{\substack{\text{macroscopic} \\ \text{current}}} + \underbrace{I(t)}_{\substack{\text{noisy} \\ \text{current}}} = -\frac{Q}{RC}$$

For the noisy current source we have thus

$$I(t) = -L(t)$$

and the stochastic description of this noisy current is identical with $L(t)$:

$$\langle I(t) \rangle = 0$$

$$\langle I(t)\, I(t') \rangle = \frac{2KT}{R}\, \delta(t-t')$$

Alternatively one may say that the noisy source is a "voltage source"

$$V_R = \left(i + I(t)\right) R$$

$$V_R = \underbrace{iR}_{\substack{\text{macroscopic} \\ \text{voltage on the} \\ \text{resistor}}} + \underbrace{I(t)\,R}_{\substack{\text{noisy voltage on} \\ \text{the resistor}}}$$

$$V_R = iR + \underbrace{V(t)}_{\text{noisy voltage}}$$

Because $V(t) = I(t)R = -L(t)R$, the stochastic properties of $V(t)$ are:

$$\langle V(t) \rangle = 0$$

$$\langle V(t)\, V(t') \rangle = 2kTR\, \delta(t-t')$$

These are merely other ways of describing the same facts, but in practice it is convenient to be able to insert the noise sources into a network, just like any other source (i.e. battery), without having to write first the equations for the whole network (see Van Kampen's book STOCHASTIC PROCESSES IN PHYSICS AND CHEMISTRY).

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THERMAL AGITATION OF ELECTRICITY IN CONDUCTORS

By J. B. Johnson

Abstract

Statistical fluctuation of electric charge exists in all conductors, producing random variation of potential between the ends of the conductor. The effect of these fluctuations has been measured by a vacuum tube amplifier and thermocouple, and can be expressed by the formula $\bar{I}^2=(2kT/\pi)\int_0^\infty R(\omega)|Y(\omega)|^2d\omega$. I is the observed current in the thermocouple, k is Boltzmann's gas constant, T is the absolute temperature of the conductor, $R(\omega)$ is the *real* component of impedance of the conductor, $Y(\omega)$ is the transfer impedance of the amplifier, and $\omega/2\pi=f$ represents frequency. *The value of Boltzmann's constant* obtained from the measurements lie near the accepted value of this constant. *The technical aspects of the disturbance* are discussed. In an amplifier having a range of 5000 cycles and the input resistance R the power equivalent of the effect is $\bar{V}^2/R=0.8\times10^{-16}$ watt, with corresponding power for other ranges of frequency. The least contribution of *tube noise* is equivalent to that of a resistance $R_c=1.5\times10^5 i_p/\mu$, where i_p is the space current in milliamperes and μ is the effective amplification of the tube.

IN TWO short notes[1] a phenomenon has been described which is the result of spontaneous motion of the electricity in a conducting body. The electric charges in a conductor are found to be in a state of thermal agitation, in thermodynamic equilibrium with the heat motion of the atoms of the conductor. The manifestation of the phenomenon is a fluctuation of potential difference between the terminals of the conductor which can be measured by suitable instruments.

The effect is one of the causes of that disturbance which is called "tube noise" in vacuum tube amplifiers.[2] Indeed, it is often by far the larger part of the "noise" of a good amplifier. When such an amplifier terminates in a telephone receiver, and has a high resistance connected between the grid and filament of the first tube on the input side, the effect is perceived as a steady rustling noise in the receiver, like that produced by the small-shot (Schrot) effect under similar circumstances. The use of a thermocouple or rectifier in place of the telephone receiver allows reasonably accurate measurements to be made on the effective amplitude of the disturbance.

It had been known for some time among amplifier technicians that the "noise" increases as the input resistance is made larger. A closer study of this phenomenon revealed the fact that a part of the noise depends on the resistance alone and not on the vacuum tube to which it is connected. The true nature of the effect being then suspected, the temperature of the re-

[1] Johnson, Nature **119**, p. 50, Jan. 8, 1927; Phys. Rev. **29**, p. 367 (Feb. 1927).

[2] The possibility that under certain conditions the heat motion of electricity could create a measurable disturbance in amplifiers has been recognized on theoretical grounds by W. Schottky (Ann. d. Phys. **57**, 541 (1918)). Schottky considered the special case of a resonant circuit connected to the input of a vacuum tube, and concluded that there the effect would be so small as to be masked by the small-shot effect in the tube.

all about the same resistance, and this was repeated with another resistance value.

Measurements and Results

A considerable part of the work consisted of comparative measurements in which the characteristics of the amplifier did not need to be known. In these circumstances only the maximum amplification was determined. It was convenient in such cases to think of the resistance as impressing on the amplifier a mean-square potential $\overline{V}^2$. By this method of comparison was determined the fact that the phenomenon is independent of the material and shape of the resistance unit and of the mechanism of the conduction,[6] but does depend on the electrical resistance. A few of the results are reproduced in Fig. 4. They are expressed in terms of $\overline{V}^2$, the apparent mean-square potential fluctuation, plotted against the resistance component $R(\omega)$. The points lie close to a straight line. The quantity $W=\overline{V}^2/R(\omega)$, which may be

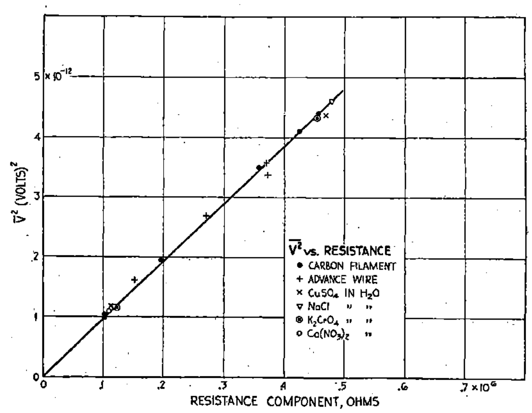

Fig. 4. Voltage-squared vs. resistance component for various kinds of conductors.

called the power equivalent of the effect, is independent of all the variables so far considered, including the electrical resistance itself.

The effect of the shunt capacity C across the conductor is shown in Fig. 5. In this case the abscissae are the measured values of resistance R_0, the circles marking the observed values of the apparent $\overline{V}^2$. These values of $\overline{V}^2$ reach a maximum and then actually decrease as the resistance is indefinitely increased. Obtaining a factor of proportionality K from the initial slope of the curve and using the measured value of C and ω for the calculation of $\overline{V}^2 = KR(\omega) = KR_0/(1+\omega^2C^2R_0^2)$, the expected values of $\overline{V}^2$ were calculated. These are represented by the curve in Fig. 5. The course of the calculated curve agrees well with that for the observed points. The agreement was also verified by using a fixed resistance and adding known shunt capacities up to as high as 60,000 mmf.

[6] Resistances such as thermionic tubes and photoelectric cells, perhaps all resistances not obeying Ohm's law, are exceptions to this rule. In these the statistical conditions are different.

The effect of the temperature of the resistance element was studied chiefly by the same comparative method that was used for the varied resistances. The experiments were done on the Advance wire resistance and on carbon filament resistances over the temperature range from $-180°C$ (liquid air) to $100°C$ (boiling water). They were also done on two liquid resistances made of alcohol and sulphuric acid, covering a range of tempera-

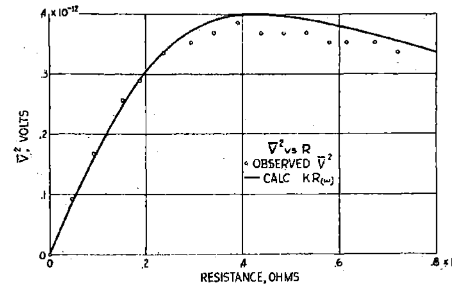

Fig. 5. Voltage-squared vs. resistance with fixed shunt capacity; frequency 635 p.p.s., capacity 577 mmf.

tures from $-72°C$ to $90°C$. The resistance values used in the computations were those measured at the various temperatures. Advance wire and carbon filaments changed very little in resistance over the temperature range used for them, while that of the liquid elements changed tremendously, increasing for lower temperatures. In all cases, however, the virtual power $\overline{V}^2/R(\omega)$

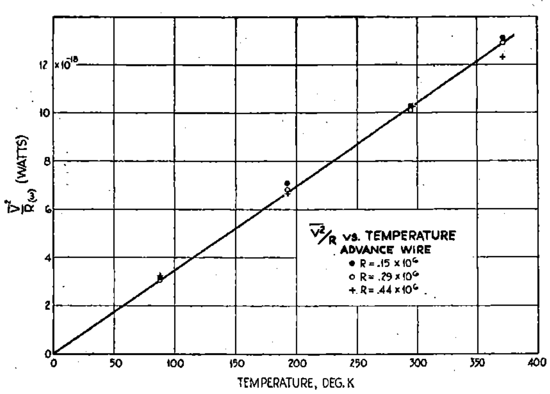

Fig. 6. Apparent power vs. temperature, for Advance wire resistances.

was proportional to the absolute temperature of the resistance element. Fig. 6 shows graphically the results for the three Advance wire resistances. The other resistances gave values falling closely on the same straight line as these.

There is finally to be recounted the verification of Eqs. (1) and (3) as they involve the electrical properties of the measuring system. The method of obtaining the frequency characteristic of the amplifier, the graphic integration of this curve and the corrections that were applied, have already been described. For these determinations the amplifier was altered in various ways, but usually by changing the resonant circuit forming one of the interstage couplings, or by replacing this circuit by the band-pass filter. The natural frequency of the resonant circuit was changed, by means of the inductance or condenser, over the range of 300 to 2000 periods per second. The sharpness of resonance was varied by changing either the resistance in the resonant circuit or the inductance in series with this circuit and the tube. The area under the characteristic curve could thus be made larger or smaller over a considerable range.

The input resistance element, in all of this work, was kept at only slightly above room temperature. It was of the "grid leak" type for most of those measurements in which the resonant circuit was employed, while the Advance

TABLE I. *Determination of Boltzmann's Constant.*

No.	f	T	$R(\omega)$	$\int_0^\infty \|Y(\omega)\|^2 d\omega$	$\int_0^\infty \|Y(\omega)\|^2 R(\omega) d\omega$	ΔS	$\overline{I^2}$	k
	p.p.s.	°K	$\times 10^{-6}$	$\times 10^{-10}$	$\times 10^{-16}$	%	$\times 10^{6}$	$\times 10^{16}$
1	1010	298	.526	.213		12.0	2.7	1.27
2	2023	"	.470	.272		26.2	2.8	1.15
3	1418	"	.508	.361		15.6	3.8	1.09
4	"	"	"	.188		42.3	1.8	.99
5	295	"	.548	.252		3.3	3.1	1.18
6	"	"	"	.202		15.8	2.0	.95
7	302	"	"	.221		12.4	2.7	1.18
8	"	"	"	.195		15.3	2.3	1.13
9	653	"	.541	.747		6.8	10.4	1.14
10	"	"	"	.645		12.9	8.6	1.30
11	1418	"	.508	.286		18.5	3.5	1.26
12	"	"	"	.161		41.3	1.7	1.09
13	1465	"	.505	1.93		18.7	21.2	1.14
14	"	"	"	1.75		20.3	19.1	1.14
15	635	"	.541	.594		8.8	8.9	1.46
16	"	"	"	.139		35.0	2.1	1.47
17	"	"	"	.597		10.9	7.8	1.28
18*	643	295	.44±		.439	0	11.0	1.38
19	645	297	"		.396	0	11.1	1.49
20	1830	301	"		.913	0	19.8	1.13
21	500–1000	299	"		.831	0	25.9	1.64
22	"	300	"		.662	0	21.5	1.70
23	"	300	"		.832	0	26.0	1.63

* The resonance curve for this determination is that reproduced in Fig. 3.

wire resistance was used as input element in connection with the band-pass filter. The resistance and capacity of each was measured accurately *in situ*, the resistance being of the order of one-half megohm.

The results will be presented in terms of the value of Boltzmann's constant k, as calculated from the data according to Eqs. (1) or (3). The last column of Table I contains these calculated values, while in the other columns of the table are indicated some of the experimental conditions involved in each determination. The columns of the table indicate, in order,

THERMAL AGITATION OF ELECTRIC CHARGE IN CONDUCTORS*

BY H. NYQUIST

ABSTRACT

The electromotive force due to thermal agitation in conductors is calculated by means of principles in thermodynamics and statistical mechanics. The results obtained agree with results obtained experimentally.

DR. J. B. JOHNSON[1] has reported the discovery and measurement of an electromotive force in conductors which is related in a simple manner to the temperature of the conductor and which is attributed by him to the thermal agitation of the carriers of electricity in the conductors. The work to be resported in the present paper was undertaken after Johnson's results were available to the writer and consists of a theoretical deduction of the electromotive force in question from thermodynamics and statistical mechanics.[2]

Consider two conductors each of resistance R and of the same uniform temperature T connected in the manner indicated in Fig. 1. The electromotive force due to thermal agitation in conductor I causes a current to be set up in the circuit whose value is obtained by dividing the electromotive force by $2R$. This current causes a heating or absorption of power in conductor II, the absorbed power being equal to the product of R and the square of the current. In other words power is transferred from conductor I to conductor II. In precisely the same manner it can be deduced that power is transferred from conductor II to conductor I. Now since the two conductors are at the same temperature it follows directly from the second law of thermodynamics that the power flowing in one direction is exactly equal to that flowing in the other direction. It will be noted that no assumption has been made as to the nature of the two conductors. One may be made of silver and the other of lead, or one may be metallic and the other electrolytic, etc.

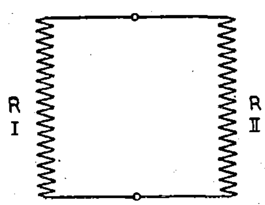

Fig. 1.

It can be shown that this equilibrium condition holds not only for the total power exchanged by the conductors under the conditions assumed, but also for the power exchanged within any frequency. For, assume that this is not so and let A denote a frequency range in which conductor I delivers more power than it receives. Connect a non-dissipative network between the two conductors so designed as to interfere more with the transfer of energy

* A preliminary report of this work was presented before the Physical Society in February, 1927.

[1] See preceding paper.

[2] Cf. W. Schottky, Ann. d. Physik 57, 541 (1918).