The Langevin Equation and Harmonic Analysis

$$m \frac{dv}{dt} = - m\gamma v + X(t)$$

with $m\gamma = 6\pi a \eta$ (Stokes)

$X(t)$ is a Gaussian noise with

$$\langle X(t) \rangle = 0$$

$$\langle X(t)\, X(t') \rangle = \Gamma\, \delta(t-t')$$

Goal

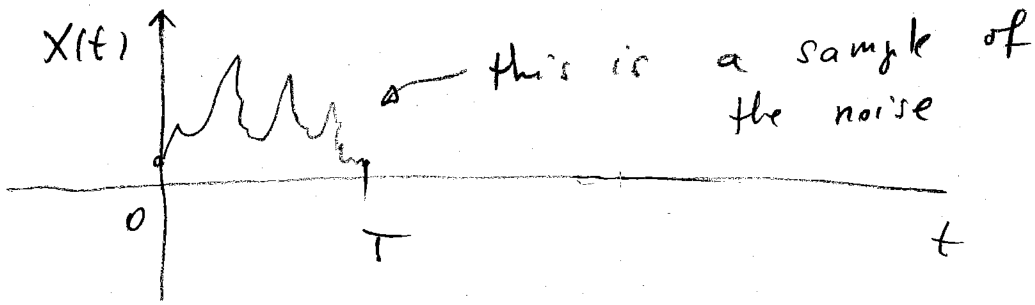
Find $\langle v(t) \rangle$

$\langle v(t)\, v(t') \rangle$

Method

Express the variables by a superposition of oscillating functions.

Consider that we observe $X(t)$ over a
a time interval $0 \le t \le T$



then write $X(t)$ as a superposition of oscillating functions

the amplitudes a_n of the oscillations

$$X(t) = \sum_{n=-\infty}^{\infty} a_n e^{i\omega_n t}$$

where

$$\omega_n = \frac{2\pi n}{T} \qquad (n = 0, \pm 1, \pm 2, \dots)$$

the interval T

Given the sample noise $X(t)$ we can find the amplitudes a_n from

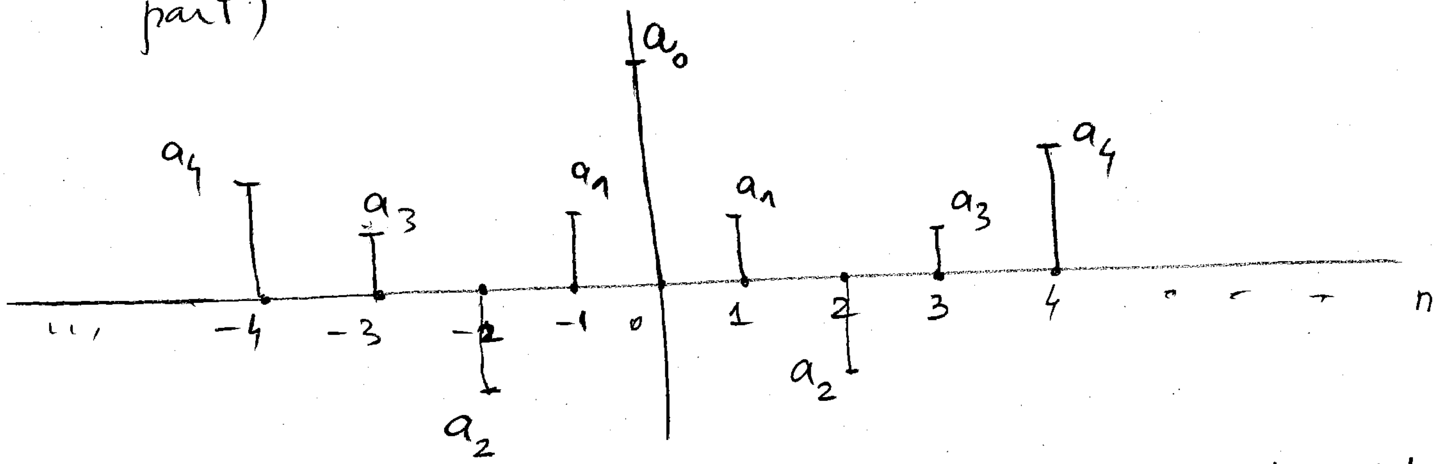
$$a_n = \frac{1}{T} \int_0^T X(t) e^{-i\omega_n t} \, dt$$

The amplitude a_n is a complex number

$$a_n = a_n' + i a_n'' , \qquad a_{-n} = a_n^* = a_n' - i a_n''$$

complex conjugate

We can plot the real (and the imaginary part)



For different noise samples $X(t)$ we will obtain a different amplitudes a_n. So the amplitudes a_n will be a stochastic process also. The aim in what follows is to find the stochastic properties of a_n, Namely, we want to find

$$\langle a_n \rangle$$

$$\langle a_n a_{n'} \rangle$$

The hope is that we can solve our initial problem ($\langle v(t) v(t') \rangle$) much more easy in terms of $\langle a_n \rangle$ and $\langle a_n a_{n'} \rangle$. The price that we have to pay is that we need to put some effort in understanding the amplitudes a_n.

From

$$a_n = \frac{1}{T}\int_0^T X(t)\, e^{-i\omega_n T}\, dt$$

we get

$$\langle a_n\rangle = \left\langle \frac{1}{T}\int_0^T X(t)\, e^{-i\omega_n T} dt \right\rangle = \frac{1}{T}\int_0^T \langle X(t)\rangle e^{-i\omega_n T} dt$$

So

$$\langle a_n\rangle = 0$$

because $\langle X(t)\rangle = 0$.

Now suppose that $\langle X(t)\rangle$ is different from zero but is a constant (i.e. is time independant)

$$\langle X(t)\rangle = \text{CONSTANT}$$

this happens when the process reached a stationary state. The particle is kept for a long time in motion, and we observe its motion long after the initial time $t=0$.

then

$$\langle a_n\rangle = \frac{1}{T}\int_0^T \text{CONSTANT}\; e^{-i\omega_n T} dt =$$

$$= \frac{1}{T}(\text{CONSTANT})\cdot \frac{1}{-i\omega_n}\underbrace{\left(e^{-i\omega_n T} - e^{-i\omega_n \cdot 0}\right)}_{e^{-i\frac{2\pi n}{\cancel{T}}\cancel{T}} - 1 = 0} =$$

$\omega_n \neq 0$

So

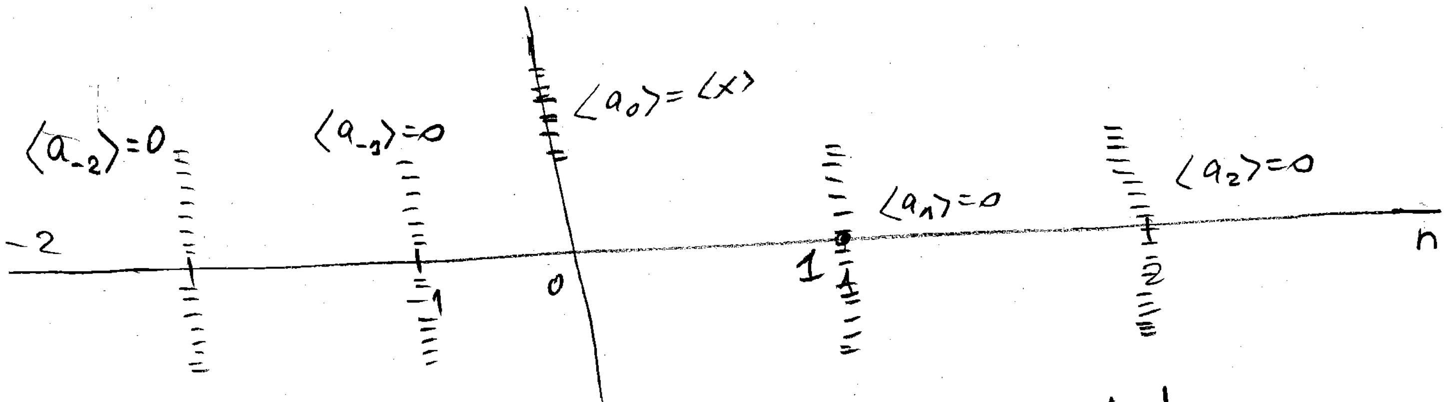
$$\langle a_n \rangle = 0 \qquad \text{for} \quad n \neq 0 \quad (\omega_n \neq 0)$$

For $n = 0$ (or $\omega_0 = 0$)

$$\langle a_0 \rangle = \frac{1}{T} \int_0^T \text{CONSTANT } e^0 \, dt = \text{CONSTANT}$$

So

$$\boxed{\begin{aligned} \langle a_0 \rangle &= \langle x \rangle \\ \langle a_n \rangle &= 0 \qquad n \neq 0 \end{aligned}}$$



Now we want the two-point correlation functions

$$\langle a_n a_{n'} \rangle$$

$$\langle a_n a_{n'} \rangle = \left\langle \left(\frac{1}{T} \int_0^T dt \; X(t) e^{-i\omega_n t} \right) \cdot \right.$$

$$\left. \frac{1}{T} \int_0^T dt' \, X(t') \, e^{-i\omega_{n'} t'} \right) \right\rangle =$$

$$= \frac{1}{T^2} \int_0^T dt \int_0^T dt' \; \langle X(t) X(t') \rangle \, e^{-i\omega_n t} \, e^{-i\omega_{n'} t'} =$$

$$= \frac{1}{T^2} \int_0^T dt \int_0^T dt' \; \Gamma \, \delta(t-t') \, e^{-i\omega_n t} \, e^{-i\omega_{n'} t'} =$$

$$= \frac{1}{T^2} \int_0^T dt \; \Gamma \; e^{-i\omega_n t} \; \underbrace{e^{-i\omega_{n'} t}}_{} . \qquad =$$

because the integration over t' with $\delta(t-t')$ changes t' into t in $e^{-i\omega_{n'} t'}$

$$= \frac{1}{T^2} \; \frac{\Gamma}{-i(\omega_n + \omega_{n'})} \left(e^{-i(\omega_n + \omega_{n'})T} - 1 \right) =$$

$$= 0 \quad \text{if } \omega_n + \omega_{n'} \neq 0$$

If $\omega_n + \omega_{n'} = 0$ the last integral is $\dfrac{\Gamma}{T}$

So

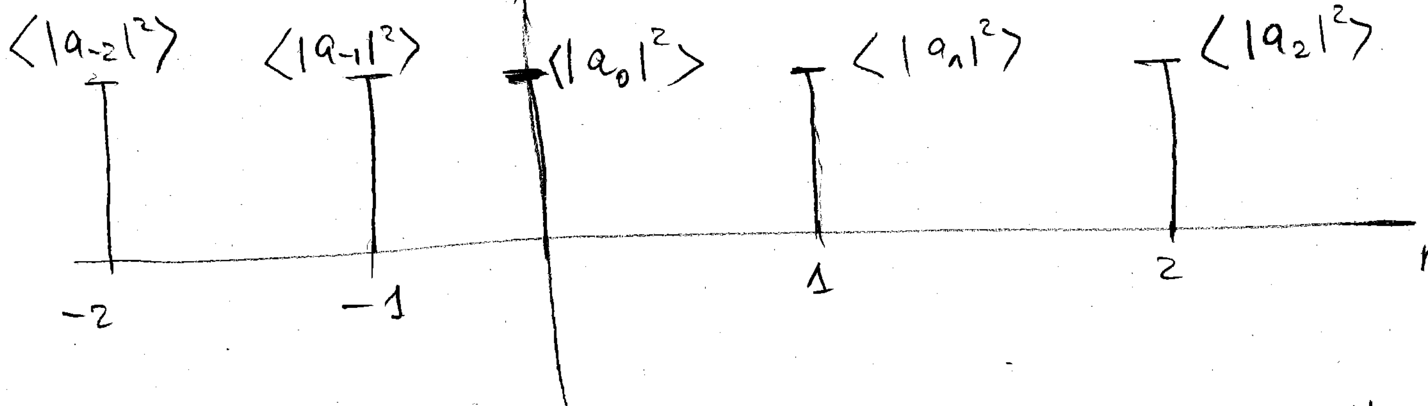
$$\langle a_n a_{n'} \rangle = \begin{cases} 0 & n \neq -n' \\ \dfrac{\Gamma}{T} & n = -n' \end{cases}$$

But $a_{-n} = a_n^*$ so for $n = -n'$

$$\langle a_n a_{n'} \rangle = \langle a_n a_n^* \rangle = \langle |a_n|^2 \rangle$$

the average of the strength of the amplitude a_n.

Strength of the amplitude (Intensity)



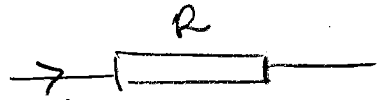
$$I_n = \langle |a_n|^2 \rangle = \frac{\Gamma}{T}$$

the intensity of the Fourier component

It is the same for all the FOURIER components

From here the name WHITE in white noise

The intensity I_n is proportional with the <u>power</u> of the Fourier component. Think of a current $a_n e^{i\omega n t}$ through a resistor R



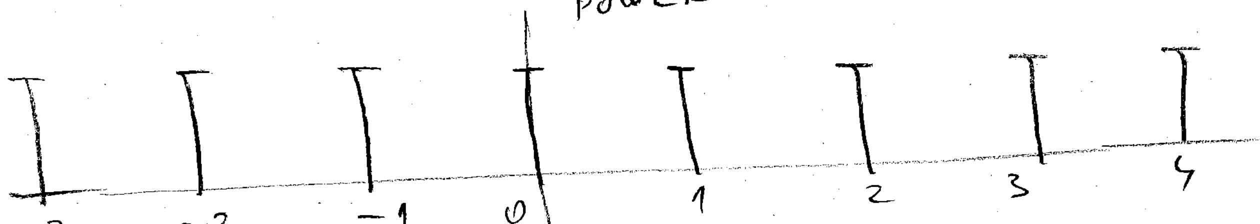
$\text{Current} = a_n e^{i\omega n t}$

the power dissipated on the resistor R is

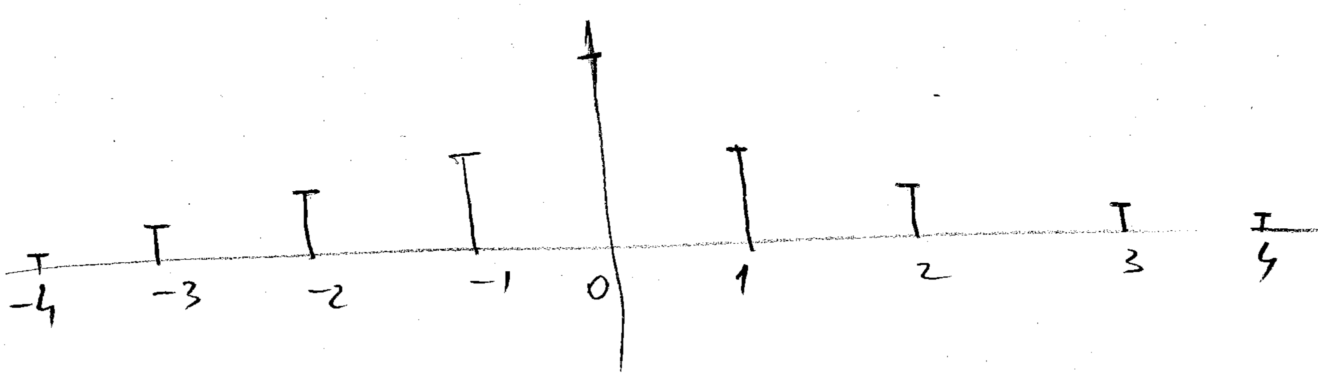
$$|\text{Current}|^2 \cdot R$$

which is $|a_n|^2 R$, so the name POWER SPECTRUM for I_n

POWER SPECTRUM



A POWER SPECTRUM which is <u>not</u> white looks like



for example

When the observation time T becomes very large ($T \to \infty$) the frequencies $\omega_n = \frac{2\pi n}{T}$ becomes very dense. The space between two adjacent frequencies

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{2\pi}{T} \qquad (1)$$

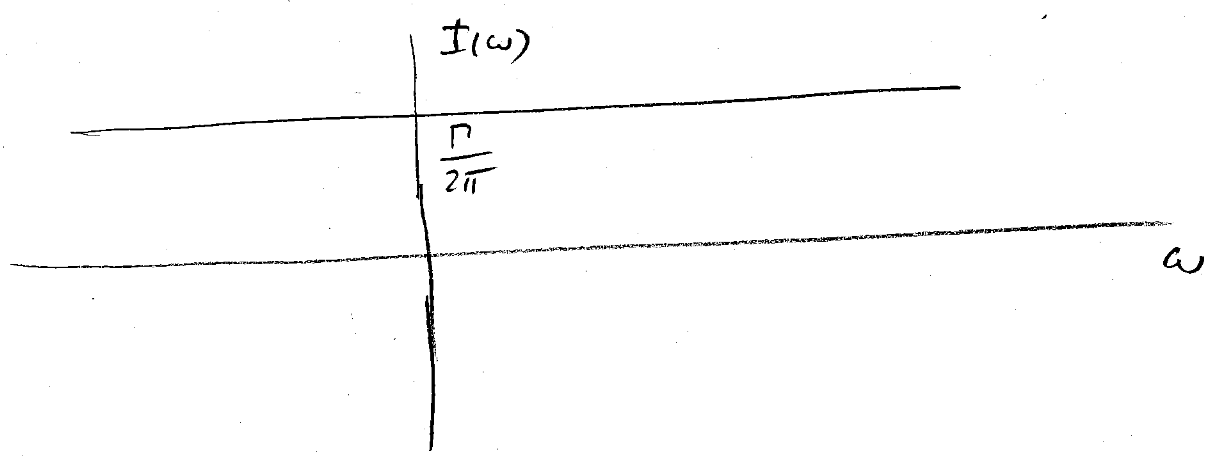
tends to zero as $T \to \infty$. Thus, for $T \to \infty$ instead of I_n we need to work with a continuous function $I(\omega)$, $\omega \in \mathbb{R}$. The definition of the POWER DENSITY is

$$I(\omega) \cdot \Delta\omega = \lim_{T \to \infty} \langle |a_n|^2 \rangle \qquad (2)$$

with $\Delta\omega = \frac{2\pi}{T}$ from (1) and $\langle |a_n|^2 \rangle = \frac{\Gamma}{T}$

we get

$$I(\omega) = \frac{\Gamma}{2\pi}$$



In general, if

$$\langle X(t_0)\, X(t_0+t)\rangle = \phi(t) \qquad (\mathrm{I})$$

(So far we studied $\phi(t)=\delta(t)$, but now we take a general $\phi(t)$. the process is stationary because $\Phi(t)$ does not depend on t_0)

then

$$\boxed{I(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(t)\, e^{-i\omega t}\,dt} \qquad (\mathrm{II})$$

In words: if we know the two point correlation function for the process $X(t)$, we cand find the two-point correlation function for the process

$$a(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(t)\,e^{-i\omega t}\,dt \qquad (\mathrm{III})$$

$$\boxed{\langle a(\omega)\, a^*(\omega')\rangle = I(\omega)\,\delta(\omega-\omega')} \qquad (\mathrm{IV})$$

Also

$$\Phi(t)=\int_{-\infty}^{\infty}I(\omega)\,e^{i\omega t}\,d\omega \qquad (\mathrm{V})$$

<u>Remark</u> (I), (II), (III), (IV) and (V) is the Wiener-Khintchine method or theorem

Practical application

$$m\dot{v}(t) = -m\gamma v(t) + X(t) \qquad (3)$$

with $X(t)$ a white noise

$$I(\omega) = I_X = \text{CONSTANT} \quad \langle X(t_1)X(t_2)\rangle = 2\pi I_X \delta(t_1-t_2)$$

In terms of $\tilde{v}(\omega)$

$$v(t) = \int_{-\infty}^{\infty} \tilde{v}(\omega)\, e^{i\omega t}\, dt$$

the equation (3) becomes

$$m(i\omega)\tilde{v}(\omega) = -m\gamma\tilde{v}(\omega) + a(\omega) \qquad (4)$$

with

$$a(\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-i\omega t}\, dt$$

So from (4)

$$\boxed{\tilde{v}(\omega) = \frac{1}{i\omega+\gamma}\,\frac{a(\omega)}{m}} \qquad (5)$$

So

$$|\tilde{v}(\omega)|^2 = \frac{1}{i\omega+\gamma}\,\frac{a(\omega)}{m}\cdot\frac{1}{-i\omega+\gamma}\,\frac{a^*(\omega)}{m}$$

$$|\tilde{v}(\omega)|^2 = \frac{1}{\omega^2+\gamma^2}\,\frac{|a(\omega)|^2}{m^2}$$

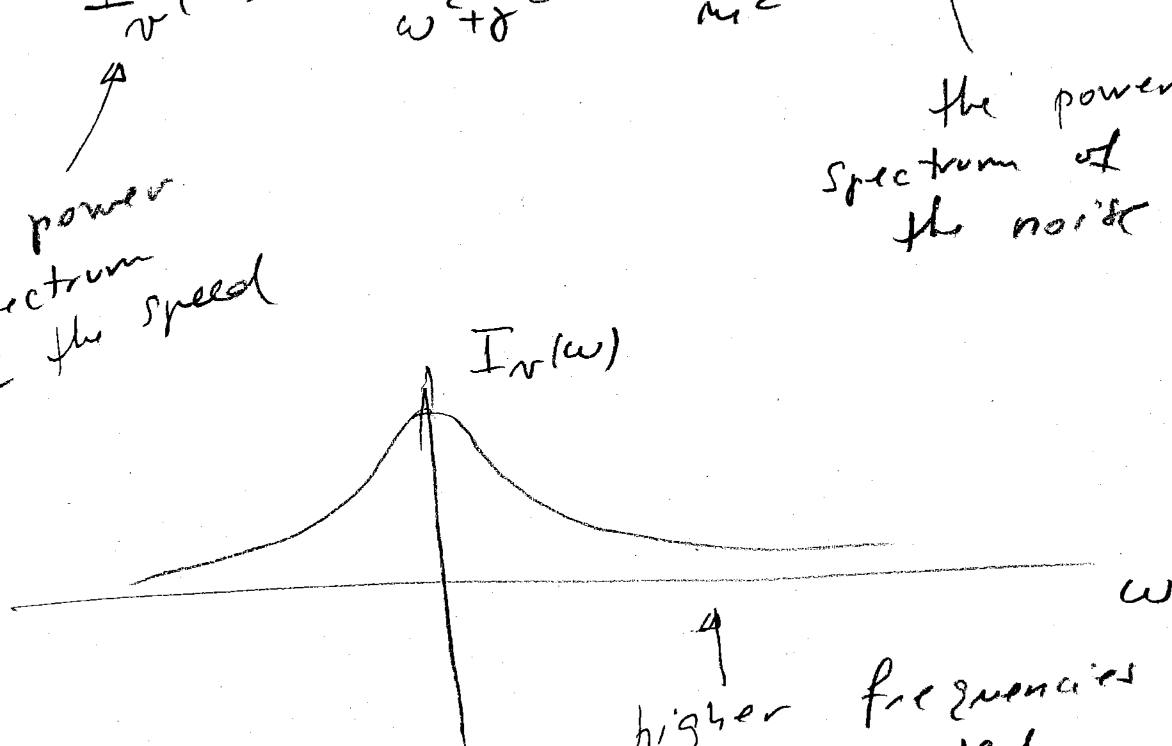
$$\langle|\tilde{v}(\omega)|^2\rangle = \frac{1}{\omega^2+\gamma^2}\,\frac{\langle|a(\omega)|^2\rangle}{m^2}$$

So

$$I_v(\omega) = \frac{1}{\omega^2+\gamma^2}\,\frac{I(\omega)}{m^2}$$

the power spectrum of the speed

the power spectrum of the noise



higher frequencies are atinuated

Now the two-point correlation function for the speed is in general

$$\phi_v(t) = \int_{-\infty}^{\infty} I_v(\omega)\, e^{i\omega t}\, d\omega$$

For our specific problem

$$\phi_v(t) = \int_{-\infty}^{\infty} \frac{1}{\omega^2+\gamma^2}\,\frac{I(\omega)}{m^2}\, e^{i\omega t}\, d\omega =$$

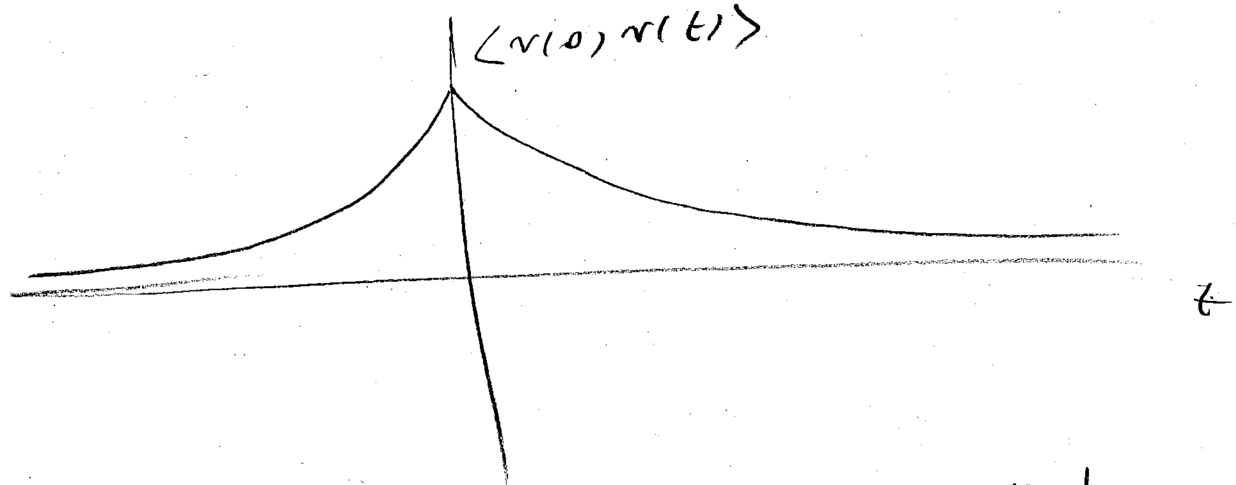
white noise

$$= \int_{-\infty}^{\infty} \frac{1}{\omega^2+\gamma^2}\,\frac{I_x}{m^2}\, e^{i\omega t}\, d\omega = \frac{\pi I_x}{m^2\gamma}\, e^{-\gamma|t|}$$

So, from $\phi_v(t) = \langle v(t_1) v(t_1 + t) \rangle$

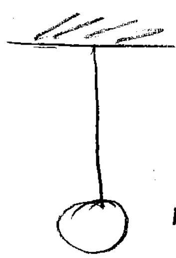
we get

$$\boxed{\langle v(t_1) v(t_2) \rangle = \frac{\pi I_x}{m^2 \gamma} e^{-\gamma |t_1 - t_2|}}$$



Brownian harmonic oscillator

$$m \frac{d^2 x}{dt^2} + m\gamma \frac{dx}{dt} + m \omega_0^2 x = \underbrace{R(t)}_{\text{noise}}$$



Torsional oscillation of a small mirror suspended in a dilute gas.

The power spectrum of the noise

$$I_R(\omega) = \langle |a(\omega)|^2 \rangle$$

is given. Find the power spectrum of $x(t)$.

Solution

$$x(t) = \int_{-\infty}^{\infty} d\omega \, e^{i\omega t} \, \tilde{x}(\omega)$$

then we get an equation for $\tilde{x}(\omega)$

$$m (i\omega)^2 \tilde{x}(\omega) + m\gamma (i\omega) \tilde{x}(\omega) + \omega_0^2 \tilde{x}(\omega) = a(\omega)$$

with

$$R(t) = \int_{-\infty}^{\infty} d\omega \, e^{i\omega t} \, a(t)$$

So

$$\tilde{x}(\omega) = \frac{1}{-\omega^2 + \gamma i\omega + \omega_0^2} \frac{a(\omega)}{m}$$

With

$$I_x(\omega) = \langle |\tilde{x}(\omega)|^2 \rangle$$

we get

$$I_x(\omega) = \frac{1}{|\omega_0^2 - \omega^2 + i\gamma\omega|^2} \frac{I_R(\omega)}{m^2}$$

$$\boxed{I_x(\omega) = \frac{1}{(\omega_0^2 - \omega^2) + \gamma^2\omega^2} \frac{I_R(\omega)}{m^2}}$$

$$\phi_x(t) = \int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\, I_R(\omega)$$

white noise

$$\phi_x(t) = \int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \frac{1}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \frac{I_R}{\mu^2}$$

$$\boxed{\phi_x(t) = \frac{\pi I_R}{\mu^2 \gamma \omega_0^2}\left(\cos\omega_1 t + \frac{\gamma}{2\omega_1}\sin\omega_1 t\right) e^{-\frac{\gamma t}{2}}}$$

$t > 0$

with

$$\omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

For $t \to \infty$ $\qquad \phi_x(t) \to 0$

$t \to 0$ $\qquad \phi_x(0) \to \dfrac{\pi I_R}{\mu^2 \gamma \omega_0^2}$

From $\phi_x(t) = \langle x(t_1) \cdot x(t_1+t)\rangle$

we get $\phi_x(0) = \langle [x(t_1)]^2 \rangle = \dfrac{\pi I_R}{\mu^2 \gamma \omega_0^2}$