More about Langevin's noise

The Langevine equation

random force

$$(1) \qquad m \frac{d^2x}{dt^2} = -6\pi\mu r \frac{dx}{dt} + X$$

Denote $\quad \gamma = +6\pi\mu r/m$

and $\quad v(t) = \dfrac{dx}{dt}$

(Langevin's notation was $\xi = \dfrac{dx}{dt}$ but we will use v for velocity)

So (1) becomes

X depends on time

$$(2) \qquad \dot{v}(t) = -\gamma v(t) + \frac{1}{m} X(t)$$

If $X(t) = 0$ the solution to (2) is

$$v(t) = v(t_0)\, e^{-\gamma(t-t_0)}$$

For $X(t) \neq 0$ we need to add a term dependant on $X(t)$

$$(3) \qquad v(t) = v(t_0)\, e^{-\gamma(t-t_0)} + \int_{t_0}^{t} e^{-\gamma(t-t')} \frac{X(t')}{m}\, dt'$$

Solution (3) to equation (2) can be obtained using the method of <u>variation of parameters</u>. This method is presented in the book "Ordinary Differential Equations" by E.L. Ince (please see the handouts).

In what follows you will see a physical interpretation of the solution (3).

I. The Equation

$$\dot{v}(t) = -\gamma v(t) + \frac{1}{\mu} X(t) \qquad (4)$$

has the following property:

if $v_1(t)$ is a solution and $v_2(t)$ is another solution, then $v_1(t)+v_2(t)$ is also a solution (we say that the equation is linear)

II. Based on I we will find <u>the solution</u> to (4) for a $\boxed{\text{complicated}}$ $X(t)$ by adding together <u>many</u> solutions that correspond to $\boxed{\text{simple}}$ $X(t)$.

III. A simple $X(t)$ is $X(t) = 0$

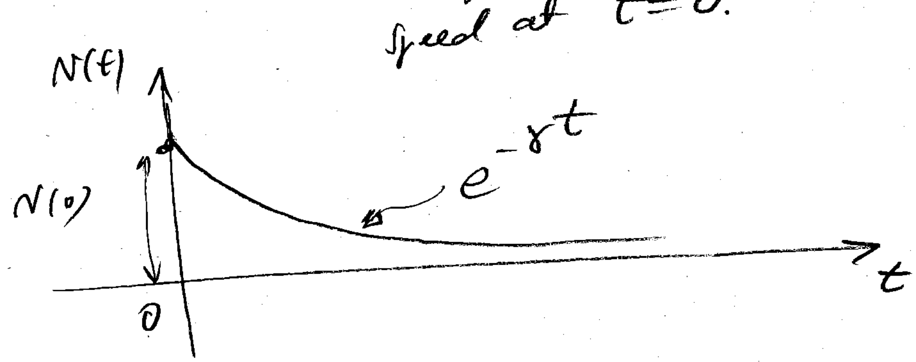
then

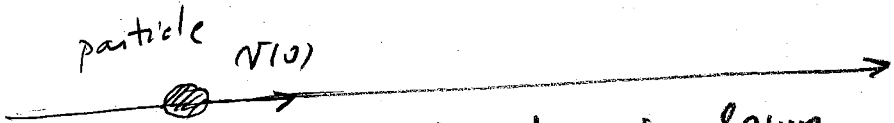
$$\dot{v}(t) = -\gamma v(t)$$

and the solution is

$$v(t) = v(0)\, e^{-\gamma t}$$

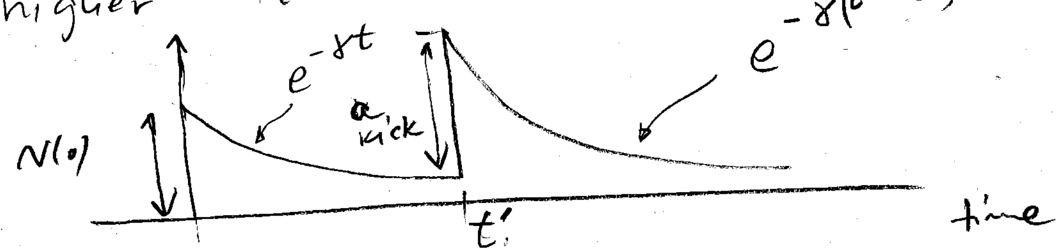
speed at $t=0$.





The particle slowes down as it moves to the right

IV. At a later time t', the particle receives a kick so its speed gets higher in an instant

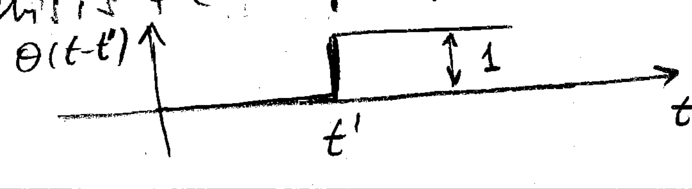


We can write then

$$(5) \quad v(t) = v_0\, e^{-\gamma t} + v'\, \theta(t-t')\, e^{-\gamma(t-t')}$$

this is the step function

that show a number the intensity of the kick in speed



$\overline{V}$. Now we will find the accelaration of the speed from $\overline{IV}$; from accelaration we will find the force that created the instantaneous kick.

$$\dot{v}(t) = -\gamma v_0 e^{-\gamma t} + v' \delta(t-t') \quad e^{-\gamma(t-t')} +$$

Dirac's delta because

$$\frac{d}{dt}\theta(t) = \delta(t)$$

$$+ v' \theta(t-t') \left(-\gamma e^{-\gamma(t-t')}\right)$$

So

$$\dot{v}(t) = -\gamma v(t) + v' \delta(t-t') e^{-\gamma(t-t')}$$

Using the property that

$$\delta(t-t') f(t) = f(t')$$

for every function $f(t)$, we get

$$\boxed{\dot{v}(t) = -\gamma v(t) + v' \delta(t-t')}$$

Compare with

$$\dot{v}(t) = -\gamma v(t) + \frac{X(t)}{m}$$

and get

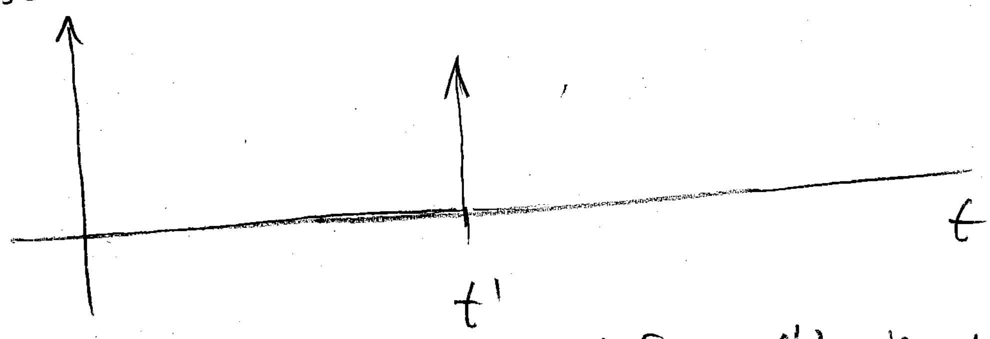
$$\boxed{X(t) = v' m \, \delta(t-t')}$$

The product $v'm$ is the intensity of the instantaneous force that is applied at the time t'. The force is instantaneous because it is a Dirac's delta function.

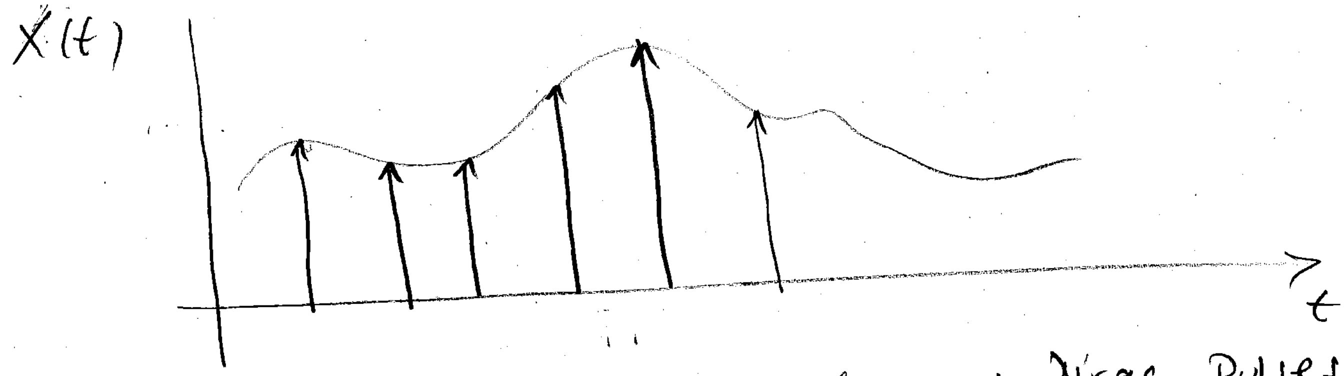
$$X(t) = x' \delta(t-t')$$

$x' = v'm$

Dirac Pulse Force



VI The instantenous force $x' \delta(t-t')$ is usefull only if we can represent <u>any</u> force $X(t)$ as a <u>sum</u> of <u>many</u> (even if infinitely many) superpositions of instantaneouses forces.



$\boxed{\text{Fig. 1}}$. This is a picture of many Dirac Pulses, with different intensities. Each pulse is located at a <u>different</u> time point.

A property of the Dirac's delta function is

$$\int_{-\infty}^{\infty} f(t')\,\delta(t-t') = f(t) \qquad (*)$$

The property (*) can be visualized as in Fig. 1. That is, the global function $f(t)$ is a superposition of Dirac pulses, each pulse being located at a different time t'; you need to sum up (integrate) along the locations of these pulses.

$\overline{\text{VII}}$ Now you are ready to find the solution of $\dot{v}(t) = -\gamma v + \frac{1}{\mu} X(t)$

for a general $X(t)$.

First you split the force in many Dirac Pulses

$$X(t) = \int_{-\infty}^{\infty} X(t')\,\delta(t-t')\,dt'$$

One pulse of intensity $X(t')$!

Dirac Pulse at t' $= X(t')\,\delta(t-t')$

will modify the speed according to (5) from step $\underline{\overline{IV}}$

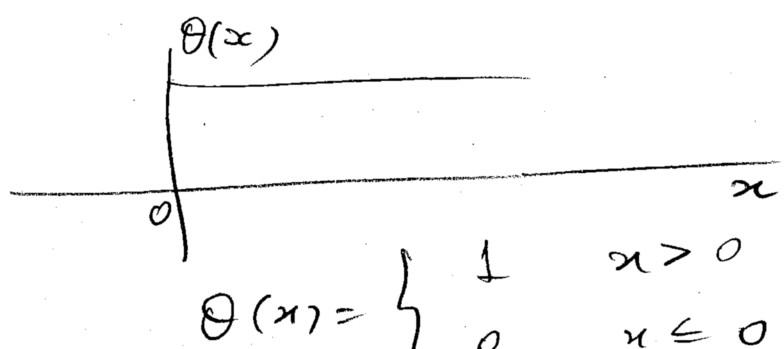
$$v(t) = v_0 e^{-\gamma t} + \frac{1}{m} X(t')\,\theta(t-t')\,e^{-\gamma(t-t')}$$

Now add (integrate) the influences of all pulses

$$v(t) = v_0 e^{-\gamma t} + \int_0^{\infty} \frac{1}{m} X(t')\,\theta(t-t')\,e^{-\gamma(t-t')}\,dt$$

Now $\theta(t-t') = 0$ if $t' > t$

Remember



$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

So

$$(6) \quad v(t) = v_0 e^{-\gamma t} + \int_0^t \frac{1}{\mu} X(t') e^{-\gamma(t-t')} dt$$

because of $\theta(t-t') = 0$ for $t' > t$.

Also $\theta(t-t')$ is out of the formula because it's equal to 1 when it is not zero.

<u>Remark</u>

The integral $\int_0^t \ldots$, goes up to t because only <u>past</u> kicks $(t' < t)$ can have an influence on the speed at time t ($v(t)$). You cannot have an influence on $v(t)$ from a <u>future</u> kick ($t' > t$) on $v(t)$.

<u>Conclusion</u> If instead of the reference $t = 0$ as our starting time, we use a reference $t = t_0$ as our starting time, the speed is

$$\boxed{v(t) = v_0 e^{-\gamma(t-t_0)} + \int_{t_0}^t \frac{1}{\mu} X(t') e^{-\gamma(t-t')} dt}$$