Fokker - Planck Equation

$$\frac{d}{dt} \underbrace{y(t)}_{m\text{-component vector}} = \underbrace{b(t, y)}_{m\text{-component vector}} + \underbrace{\Gamma(t, x)}_{m \times n \text{ vector}} \underbrace{\xi(t)}_{\text{white Gaussian noise; a vector with } n \text{ components}}$$

Example

$$m \ddot{x} = -\gamma \dot{x} + \Gamma \xi(t) \qquad (1)$$

With $\dot{x} = v$ (velocity)

the Newton's equation (1) is written as

$$\dot{x} = v \quad +$$
$$\dot{v} = -\frac{\gamma}{m} v + \frac{\Gamma}{m} \xi(t)$$

Introduce the vector

$$y = \begin{bmatrix} x \\ v \end{bmatrix}$$

then

$$\frac{d}{dt} \underbrace{\begin{bmatrix} x \\ v \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} v \\ -\frac{\gamma}{m} v \end{bmatrix}}_{b(t,y)} + \underbrace{\begin{bmatrix} 0 \\ \frac{\Gamma}{m} \end{bmatrix}}_{\Gamma(t,y)} \underbrace{\begin{bmatrix} \xi(t) \end{bmatrix}}_{\text{one-component vector}}$$

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t)\, \xi(t') \rangle = \delta(t-t')$$

In general, for an n-component noise vector

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}$$

we have

$$\langle \xi_i \rangle = 0$$

$$\langle \xi_i(t)\, \xi_j(t') \rangle = \delta_{ij}\, \delta(t-t')$$

for $i, j = 1, 2, \ldots n$.

<u>The FOKKER-PLANCK equation</u>

for $P(y_1, y_2, \ldots, y_m, t)$

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{m} \frac{\partial}{\partial y_i}(b_i P) + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2}{\partial y_i \partial y_j}(a_{ij} P)$$

with $(a_{ij}) = \sigma \sigma^T$

If at time $t=t_0$ we know that

$$y(t_0) = z,$$

then the initial condition for the probability density in the Fokker-Planck equation is

$$P(y_1, y_2, \dots, t_0) = \prod_{i=1}^{m} \delta(y_i - z_i)$$

<u>Example continued</u>

$$b_1 = v \qquad \Gamma = \begin{bmatrix} 0 \\ \frac{\Gamma}{m} \end{bmatrix}$$

$$b_2 = -\frac{\gamma}{m} v$$

$$(a_{ij}) = \Gamma \Gamma^T = \begin{pmatrix} 0 \\ \frac{\Gamma}{m} \end{pmatrix} \begin{bmatrix} 0 & \frac{\Gamma}{m} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{\Gamma}{m}\right)^2 \end{bmatrix}$$

so $a_{11} = 0$, $a_{12} = 0$, $a_{21} = 0$, $a_{22} = \left(\frac{\Gamma}{m}\right)^2$

The Fokker-Planck equation is then an equation for $P(x, v, t)$ because $y_1 = x$ and $y_2 = v$.

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(vP) - \frac{\partial}{\partial v}\left(-\frac{\gamma}{m} v P\right) + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left[\left(\frac{\Gamma}{m}\right)^2 P\right]$$

Simplify

$$\mu \dot{v} = -\gamma v + \Gamma \xi(t)$$

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial v}\left(-\frac{\gamma}{\mu} v P\right) + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left[\left(\frac{\Gamma}{\mu}\right)^2 P\right]$$

Stationary process

$$\frac{\partial P}{\partial t} = 0$$

So

$$0 = \frac{\partial}{\partial v}(v P) + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left(\left(\frac{\Gamma}{\mu}\right)^2 P\right)$$

Result $\boxed{P(v) = e^{-c \cdot v^2}}$

c : some constant

Indeed

$$\frac{\partial}{\partial v}(P) = -2cv\, e^{-cv^2}$$

$$\frac{\partial^2}{\partial v^2}(P) = -2c\, e^{-cv^2} + 4cv^2 e^{-cv^2}$$

$$\frac{\partial}{\partial v}(vP) = \frac{\partial}{\partial v}\left(v e^{-cv^2}\right) = e^{-cv^2} - 2cv^2 e^{-cv^2}$$