On page 257, the author gives the essence of his proof using Dirac's delta function.
This is an example of a rigurous mathematical proof followed by a simple explanation of the proof.

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Spectral Analysis and Time Series

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M. B. Priestley

Department of Mathematics, University of Manchester Institute of Science and Technology

ACADEMIC PRESS
Harcourt Brace Jovanovich, Publishers
London San Diego
New York Berkeley Boston Sydney
Tokyo Toronto

$h(\omega)$ being the (non-normalized) spectral density function, (4.11.2) reduces to

$$E[|dZ(\omega)|^2] = h(\omega)\, d\omega. \tag{4.11.3}$$

The above main result, namely that (virtually) any stationary process can be represented as *a "sum" (of the form* (4.11.1)) *of sine and cosine functions with uncorrelated coefficients* is certainly one of the most important ones in the whole of the theory of stationary processes. (The restriction "virtually" is due to the fact that in the continuous parameter case the process has to be stochastically continuous—the same condition as was required for the spectral representation of the autocorrelation function given in the Wiener–Khintchine theorem.) Not only does it provide a "canonical form" for describing any stationary process, but, as pointed out above, it is crucial to the physical interpretation of power spectra. It is a fascinating result also in that it lends itself to a variety of different proofs which together reveal a rich collection of mathematical ideas. The proof which we present below is a heuristic one in the sense that here we do not concern ourselves overmuch with the mathematical "frills." However, the discussion will, we trust, illustrate the essential ideas involved in the derivation of (4.11.1). A formal statement of the result is as follows.

Theorem 4.11.1 Spectral representation of continuous parameter stationary processes *Let $\{X(t)\}$, $-\infty < t < \infty$, be a zero-mean stochastically continuous stationary process. Then there exists an orthogonal process, $\{Z(\omega)\}$, such that, for all t, $X(t)$ may be written in the form,*

$$X(t) = \int_{-\infty}^{\infty} e^{it\omega}\, dZ(\omega), \tag{4.11.4}$$

the integral being defined in the mean-square sense. The process $\{Z(\omega)\}$ has the following properties;

(i) $E[dZ(\omega)] = 0,\qquad$ *all* ω,

(ii) $E[|dZ(\omega)|^2] = dH(\omega),\qquad$ *all* ω, $\qquad$ (4.11.5)
where $H(\omega)$ is the (non-normalized) integrated spectrum of $X(t)$,

(iii) *for any two distinct frequencies,* ω, ω', $(\omega \neq \omega')$,

$$\operatorname{cov}[dZ(\omega), dZ(\omega')] = E[dZ^*(\omega)\, dZ(\omega')] = 0. \tag{4.11.6}$$

Proof. We start by considering a single realization, $X(t)$, on a *finite* interval, $-T \leqslant t \leqslant T$, and then make this realization periodic outside this interval. Thus, we define a new function, $X_T^*(t)$, by,

$$X_T^*(t) = X(t), \qquad -T \leqslant t \leqslant T,$$

$$X_T^*(t+2pT) = X^*(t), \qquad p = \pm 1, \pm 2, \ldots.$$

Then $X_T^*(t)$ is periodic, with period $2T$, and hence, according to Section 4.5 (equation (4.5.3)), may be written as a Fourier series in the form,

$$X_T^*(t) = \sum_{n=-\infty}^{\infty} A_n e^{2\pi i f_n t}, \tag{4.11.7}$$

where $f_n = n/2T$ and A_n is given by

$$A_n = \frac{1}{2T}\int_{-T}^{T} X_T^*(t)\, e^{-2\pi i f_n t}.$$

(see (4.5.5)). We may now rewrite (4.11.7) in the form

$$X(t) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} G_T(\omega_n)\, e^{i\omega_n t}\delta\omega_n, \tag{4.11.8}$$

where the function $G_T(\omega)$ is defined *for all* ω by,

$$G_T(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-T}^{T} X_T^*(t)\, e^{-i\omega t}\, dt,$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-T}^{T} X(t)\, e^{-i\omega t}\, dt, \tag{4.11.9}$$

and $\omega_n = 2\pi n/2T$, so that $\delta\omega_n = \omega_{n+1} - \omega_n = 2\pi/2T$.

Although $X_T^*(t)$ is not the same as the function $X_T(t)$ defined by (4.7.1) (recall that $X_T(t)$ was defined to be *zero* outside $(-T, T)$), the function $G_T(\omega)$ defined by (4.11.9) is exactly the same as the $G_T(\omega)$ defined by (4.7.3). Hence, if we now think of (4.11.9) as defining $G_T(\omega)$ in terms of the *process* $X(t)$, then we know from (4.7.5) that when the (non-normalized) spectral density function, $h(\omega)$, exists,

$$\lim_{T\to\infty}\left[E\left\{\frac{|G_T(\omega)|^2}{2T}\right\}\right] = h(\omega),$$

so that, as $T \to \infty$,

$$|G_T(\omega_n)| = O(\sqrt{T}) = O(1/\sqrt{\delta\omega_n}).$$

Thus, $|G_T(\omega_n)| \to \infty$ as $T \to \infty$, *but* $\{|G_T(\omega_n)|\delta\omega_n\} = O(\sqrt{\delta\omega_n}) \to 0$ as $T \to \infty$.

We now define the function

$$Z_T(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\omega} G_T(\theta)\, d\theta.$$

Then,

$$\Delta Z_T(\omega_n) = Z_T(\omega_{n+1}) - Z_T(\omega_n) \sim \frac{1}{\sqrt{2\pi}} G_T(\omega_n)\delta\omega_n,$$

relationships between the spectral theory of stationary processes and other branches of pure mathematics. We now give a brief sketch of some of these alternative proofs.

(A) ANALYTICAL APPROACH

Perhaps the most straightforward way of proving (4.11.4) rigorously is to "reverse" the argument used in the heuristic proof of Theorem 4.11.1, i.e. given the process $\{X(t)\}$, we *define* $Z(\omega)$ by (4.11.14) (this defines $Z(\omega)$ up to an additive constant). We may then show that $Z(\omega)$ is an orthogonal process (this follows fairly easily from the form of the factor multiplying $X(t)$ in the integral in (4.11.14)), and then prove that, with this definition of $Z(\omega)$, the integral on the right-hand side of (4.11.4) represents $X(t)$ in mean-square, i.e. that

$$E\left[\left|X(t)-\int_{-\infty}^{\infty}\exp(it\omega)\,dZ(\omega)\right|^2\right]=0.$$

This approach is due to Blanc-Lapierre and Fortet (1946), and further details are given in Bartlett (1955), p. 169, and Yaglom (1962), p. 36.

(B) FUNCTION-THEORY APPROACH

Cramer (1951) constructed an interesting proof using the methods of "function theory" in a Hilbert space setting. This approach has now become well established in the theory of stationary processes, and in particular Parzen (1959), developed this technique in a very lucid and elegant manner. The basic ideas may be described as follows.

We first consider the collection of all (complex valued) random variables U which have zero mean and finite variance, i.e.

$$E(U)=0, \qquad E(|U|^2)<\infty.$$

We may show that this collection forms a Hilbert space H (see Section 4.2.2) if we define the inner-product between any two random variables U, V by

$$(U, V)=E(U^*V),$$

so that the norm of U is then given by,

$$\|U\|^2=E(|U|^2).$$

(With this definition of inner product two random variables are "orthogonal" if they are uncorrelated, and it is thus consistent with the use of the term "orthogonal" in probability theory usage.) For each t, $X(t)$ is a random variable of the above type and hence belongs to H. As t varies from $-\infty$ to $+\infty$, $X(t)$ traces out a "curve" in H; let H_x denote the smallest subspace of H which contains this "curve".

Now we know from the Wiener–Khintchine theorem that (with $E[X(t)]=0$),

$$E[X^*(s)X(t)]=\int_{-\infty}^{\infty}e^{i\omega(t-s)}\,dH(\omega). \tag{4.11.27}$$

For each fixed t we may think of $\exp(i\omega t)$ as a function of ω, and henceforth we will denote this function of ω by $\phi_t(\omega)$, so that (4.11.27) can be written as,

$$E[X^*(s)X(t)]=\int_{-\infty}^{\infty}\phi_s^*(\omega)\phi_t(\omega)\,dH(\omega). \tag{4.11.28}$$

We now introduce a second Hilbert space H_ϕ, which is the space "spanned" by the family of functions, $\{\phi_t(\omega)\}$, $-\infty<t<\infty$, i.e. H_ϕ consists of all functions ϕ which may be expressed as linear combinations of the $\{\phi_t(\omega)\}$, i.e. which may be written in the form,

$$\phi(\omega)=\sum_i c_i\phi_{t_i}(\omega),$$

(the $\{c_i\}$ being constants), together with functions which are obtained as limits of such linear combinations. The inner-product between any two functions, $\phi_1(\omega)$, $\phi_2(\omega)$ in H_ϕ is defined by,

$$(\phi_1(\omega), \phi_2(\omega))=\int_{-\infty}^{\infty}\phi_1^*(\omega)\phi_2(\omega)\,dH(\omega). \tag{4.11.29}$$

We now set up a mapping, M, between elements of H_x and elements of H_ϕ which is such that, for each t,

$$M[\phi_t(\omega)]=X(t),$$

and is extended to linear combinations of the $\{\phi_t(\omega)\}$ by

$$M\left[\sum_i c_i\phi_{t_i}(\omega)\right]=\sum_i c_i M[\phi_{t_i}(\omega)].$$

This mapping, M, clearly preserves inner-products since, for any s, t,

$$(X(s), X(t))=\int_{-\infty}^{\infty}\phi_s^*(\omega)\phi_t(\omega)\,dH(\omega)=(\phi_s(\omega), \phi_t(\omega)).$$

Now for any interval, (ω_a, ω_b), define the "indicator function", $I_{\omega_a,\omega_b}(\omega)$, by

$$I_{\omega_a,\omega_b}(\omega)=\begin{cases}1, & \omega_a\leqslant\omega<\omega_b,\\ 0, & \text{otherwise.}\end{cases}$$

Let $(\omega_{-n} < \omega_{-n+1} < \ldots < \omega_0 < \ldots < \omega_{n-1} < \omega_n)$ be a subset of points on the ω-axis. The crux of the proof lies in recognizing the intuitively obvious fact that we can approximate to $\phi_t(\omega)$ by sums of the form,

$$\phi_t(\omega) \sim \sum_{j=-n}^{n-1} a_j I_{\omega_j, \omega_{j+1}}(\omega),$$

where $a_j = \phi_t(\omega_j)$. In effect, we are simply forming a "step-function" approximation to $\phi_t(\omega)$, and clearly the accuracy of the approximation will increase as we increase n and decrease the interval between successive points, ω_j, ω_{j+1}. In fact, there is a well known result in the theory of functions which states that any continuous function can be constructed as the limit of a sequence of functions, starting from step functions of the above type, and this result underlies one approach to the definition of Lebesgue integration—see Riesz and Sz. Nagy (1955). Hence, we may write,

$$\phi_t(\omega) = \lim_{n\to\infty} \sum_j a_j I_{\omega_j, \omega_{j+1}}(\omega). \tag{4.11.30}$$

Now define the process $Z(\omega)$ by writing, for any ω_a, ω_b, s.t. $\omega_a \leqslant \omega_b$,

$$Z(\omega_b) - Z(\omega_a) = M[I_{\omega_a, \omega_b}(\omega)].$$

Then $Z(\omega)$ is clearly an orthogonal process since for any two non-overlapping intervals (ω_1, ω_2), (ω_3, ω_4),

$$E[\{Z(\omega_4) - Z(\omega_3)\}^* \{Z(\omega_2) - Z(\omega_1)\}] = (I_{\omega_4, \omega_3}(\omega), I_{\omega_2, \omega_1}(\omega))$$
$$= 0, \quad \text{by (4.11.29), (cf. (4.11.18)).}$$

Also,

$$E[|Z(\omega_b) - Z(\omega_a)|^2] = \|I_{\omega_b, \omega_a}(\omega)\|^2 = \int_{\omega_a}^{\omega_b} dH(\omega)$$
$$= H(\omega_b) - H(\omega_a). \tag{4.11.31}$$

Now applying the mapping M to each side of (4.11.30) we obtain,

$$M[\phi_t(\omega)] = \lim_{n\to\infty} \sum_j a_j M[I_{\omega_j, \omega_{j+1}}(\omega)]$$
$$= \lim_{n\to\infty} \sum_j \phi_t(\omega_j)\{Z(\omega_{j+1}) - Z(\omega_j)\}. \tag{4.11.32}$$

As $n \to \infty$ and the intervals between the $\{\omega_j\}$ decreases the right-hand side of (4.11.32) converges to the Stieltjes integral,

$$\int_{-\infty}^{\infty} \phi_t(\omega)\, dZ(\omega).$$

Here you see the use of the Dirac's delta function as an explanation for the essence of a proof

But, by the definition of M, $M[\phi_t(\omega)] = X(t)$, and $\phi_t(\omega)$ is, by definition, the function $\exp(it\omega)$. Hence we finally obtain,

$$X(t) = \int_{-\infty}^{\infty} e^{it\omega}\, dZ(\omega),$$

and, on writing ω for ω_a, $\omega + d\omega$ for ω_b in (4.11.31), we have,

$$E[|dZ(\omega)|^2] = dH(\omega).$$

A full account of Cramer's original proof is given in Doob (1953) and Grenander and Rosenblatt (1957a), and a somewhat more general version of essentially the same approach is given by Parzen (1959, 1961a).

The essence of the above proof may be described fairly simply in the following way. What we are doing finally (as $n \to \infty$) is using a set of δ-functions as an orthogonal basis for the space H_ϕ, and then writing,

$$\phi_t(\omega) = \int_{-\infty}^{\infty} \phi_t(\theta)\delta(\theta - \omega)\, d\theta, \quad \text{all } \omega. \tag{4.11.32a}$$

For each ω, the mapping of $\delta(\theta - \omega)$ is, in effect, the quantity "$\{dZ(\omega)/d\omega\}$", and applying the mapping to the above representation of $\phi_t(\omega)$ immediately gives the spectral representation of $X(t)$. In fact, these ideas can be made quite precise, and a more succinct version of the above proof (which bypasses the limiting process) can be constructed as follows. First, we re-write (4.11.32a) in a more rigorous form as,

$$\phi_t(\omega) = \int_{-\infty}^{\infty} \phi_t(\theta)\, dI(\theta - \omega), \tag{4.11.32b}$$

where

$$I(\theta) = \begin{cases} 1, & \theta \geqslant 0, \\ 0, & \theta < 0, \end{cases}$$

(step function
OR
UNIT STEP)

so that $I(\theta - \omega)$ is the indicator function of the set $(-\infty, \theta)$, i.e. $I(\theta - \omega) \equiv I_{-\infty,\theta}(\omega)$, in the previous notation. (Note that if the right-hand side of (4.11.32b) is interpreted as a Lebesgue–Stieltjes integral then the result holds quite generally, we do not even require continuity of the $\phi_t(\omega)$.) Now, for each θ, define $Z(\theta)$ by

$$Z(\theta) = M[I(\theta - \omega)].$$

This is exactly the same as the previous definition of $Z(\theta)$, recalling that $I(\theta - \omega) \equiv I_{-\infty,\theta}(\omega)$. We now apply the mapping M to both sides of