ORDINARY DIFFERENTIAL EQUATIONS

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The equation becomes

$$\frac{d\eta}{d\xi}=F\left(\frac{A\xi+B\eta}{a\xi+b\eta}\right),$$

so that F is a homogeneous function of ξ, η of degree zero. The constants h, k are determinate since $Ab-aB\neq 0$.

When $Ab-aB=0$, let η be a new dependent variable defined by

$$\eta=x+By/A=x+by/a,$$

then

$$\frac{d\eta}{dx}=1+\frac{b}{a}F\left(\frac{A\eta+C}{a\eta+c}\right).$$

The variables are now separable.

Example.—

$$(3y-7x+7)dx+(7y-3x+3)dy=0.$$

The substitution

$$x=\xi+1,\ y=\eta$$

reduces the equation to

$$(3\eta-7\xi)d\xi+(7\eta-3\xi)d\eta=0.$$

It is now homogeneous; the transformation $\eta=v\xi$ changes it into

$$(7v-3)\xi dv+(7v^2-7)d\xi=0$$

or

$$\left(\frac{2}{v-1}+\frac{5}{v+1}\right)dv+\frac{7}{\xi}d\xi=0,$$

whence

$$(v-1)^2(v+1)^5\xi^7=c,$$

where c is the constant of integration, that is

$$(\eta-\xi)^2(\eta+\xi)^5=c.$$

The primitive therefore is

$$(y-x+1)^2(y+x-1)^5=c.$$

2·13. Linear Equations of the First Order.—The most general linear equation of the first order is of the type

$$\frac{dy}{dx}+\phi y=\psi,$$

where ϕ and ψ are functions of x alone. Consider first of all the homogeneous linear equation *

$$\frac{dy}{dx}+\phi y=0.$$

Its variables are separable, thus:

$$\frac{dy}{y}+\phi dx=0,$$

and the solution is

$$y=ce^{-\int\phi dx},$$

where c is a constant.

Now substitute in the non-homogeneous equation, the expression

$$y=ve^{-\int\phi dx},$$

* The term *homogeneous* is applied to a linear equation when it contains no term independent of y and the derivatives of y. This usage of the term is to be distinguished from that of the preceding section in which an equation (in general non-linear) was said to be homogeneous when P and Q were homogeneous functions of x and y of the same degree. There should be no confusion between the two usages of the term.

in which v, a function of x, has replaced the constant c. The equation becomes

$$\frac{dv}{dx}e^{-\int\phi dx}=\psi,$$

whence

$$v=C+\int\psi e^{\int\phi dx}dx.$$

The solution of the general linear equation is therefore

$$y=Ce^{-\int\phi dx}+e^{-\int\phi dx}\int\psi e^{\int\phi dx}dx,$$

and involves two quadratures.

The method here adopted of finding the solution of an equation by regarding the parameter, or constant of integration c of the solution of a simpler equation, as variable, and so determining it that the more general equation is satisfied, is a particular case of what is known as the method of *variation of parameters.**

It is to be noted that the general solution of the linear equation is linearly dependent upon the constant of integration C. Conversely the differential equation obtained by eliminating C between any equation

$$y=Cf(x)+g(x),$$

and the derived equation

$$y'=Cf'(x)+g'(x),$$

is linear.

If any particular solution of the linear equation is known, the general solution may be obtained by one quadrature. For let y_1 be a solution, then the relation

$$\frac{dy_1}{dx}+\phi y_1=\psi$$

is satisfied identically. By means of this relation, ψ can be eliminated from the given equation, which becomes

$$\frac{d}{dx}(y-y_1)+\phi(y-y_1)=0.$$

The equation is now homogeneous in $y-y_1$, and has the solution

$$y-y_1=Ce^{-\int\phi dx},$$

where C is the constant of integration.

If two distinct particular solutions are known, the general solution may be expressed directly in terms of them. For it is known that the general solution has the form

$$y=Cf(x)+g(x),$$

and any two particular solutions y_1 and y_2 are obtained by assigning definite values C_1 and C_2 to the arbitrary constant C, thus

$$y_1=C_1f(x)+g(x),$$
$$y_2=C_2f(x)+g(x),$$

and therefore

$$\frac{y-y_1}{y_2-y_1}=\frac{C-C_1}{C_2-C_1}.$$

Examples.—(i) $y'-ay=e^{mx}$ (a and m constants, $m\neq a$).

The solution of the homogeneous equation

$$y'-ay=0$$

* *Vide* § 5·23. The application of the method to the linear equation of the first order is due to John Bernoulli, *Acta Erud.*, 1697, p. 113, but the solution by quadratures was known to Leibniz several years earlier.