Delta function

Origin: Dirac Quantum Mechanics

$\boxed{\text{vectors}}$

$$f = [f_1, f_2, \ldots, f_N]$$

$$g = [g_1, g_2, \ldots, g_N]$$

orthogonal when

$$f \cdot g = 0$$

or

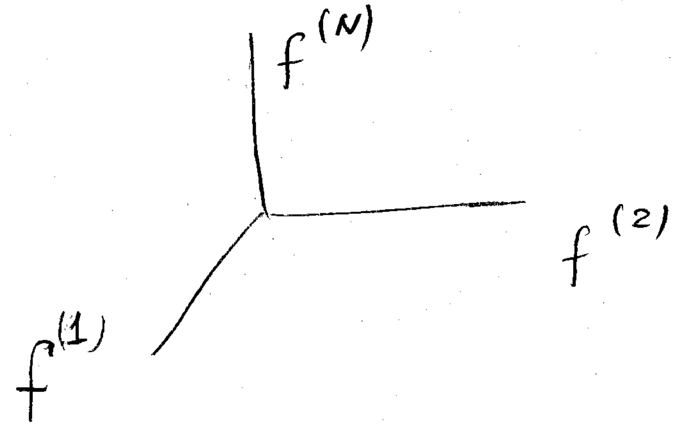
$$\sum_{m=0}^{N} f_m g_m = 0$$

unit length

$$\sum_{m=0}^{N} f_m f_m = 1$$

$$\sum_{m=0}^{N} g_m g_m = 1$$

Now you have a <u>list</u> of vectors, to specify an N-dimensional space



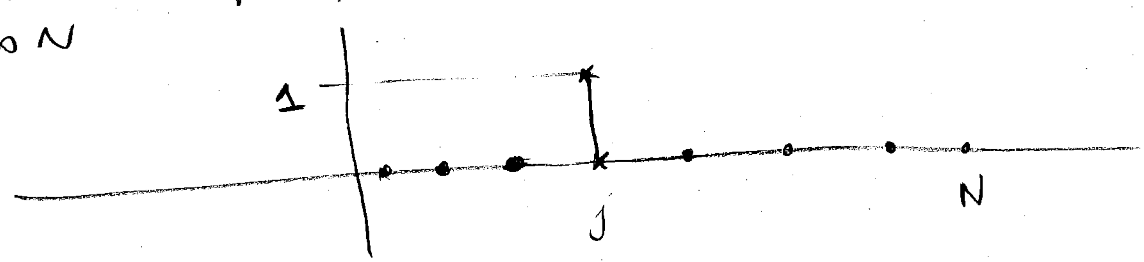
These vectors are orthogonal and have unit length

$$\left.\begin{array}{l} f^{(1)} \cdot f^{(1)} = 1 \\ f^{(1)} \cdot f^{(2)} = 0 \\ \vdots \\ f^{(N)} \cdot f^{(N)} = 1 \end{array}\right\} \text{ all } N^2 \text{ possible pairs}$$

To write short we <u>invent</u> a notation (Kronecker delta)

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

You can plot δ_{ij} if you fix j and let i run from 1 to N



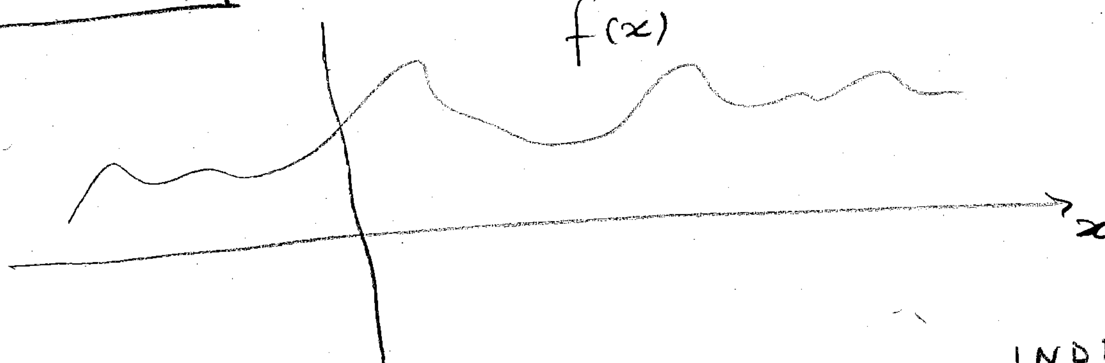
The ortogonality and the unit length can be expressed as

$$f^{(i)} \cdot f^{(j)} = \delta_{ij}$$

In components

$$\sum_{\mu=1}^{N} f_{\mu}^{(i)} f_{\mu}^{(j)} = \delta_{ij}$$

$\boxed{\text{Functions}}$



You can imagine that x is an INDEX, like μ was before, except that x runs from $-\infty$ to $+\infty$ through all values and not only discrete values like μ.

So we can imagine writing

$$f = [f_{-\infty} \cdots f_{-100.001} \cdots f_{-100.1} \cdots f_{0} \cdots f_{0.32} \cdots f_{\infty}]$$

Discrete	Continuous
Component of the vector f at position μ $\Big]$ f_μ	Component of the vector f at position x $\Big]$ f_x or $f(x)$
Scalar product of two vectors $f \cdot g = \sum_{\mu=1}^{N} f_\mu g_\mu$	Scalar product of two vectors $f \cdot g = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$
A system of N vectors $\sum_{\mu=1}^{N} f_\mu^{(i)} f_\mu^{(j)} = \delta_{ij}$ $i = 1, 2, \dots, N$ $j = 1, 2, \dots, N$	$\boxed{\text{Case } 1}$ μ becomes CONTINUOUS i and j remain DISCRETE $\int_{-\infty}^{\infty} dx\, f^{(i)}(x)\, f^{(j)}(x) = \delta_{ij}$

$\boxed{\text{Case } 2}$

μ remains DISCRETE

i and j become CONTINUOUS

$$\sum_{\mu=1}^{\infty} f_\mu(x)\, f_\mu(y) = \, ?$$

We need the continuous version of Kronecker delta

Discrete delta (Kronecker)

$$\delta_{ij} = \delta(i-j)$$

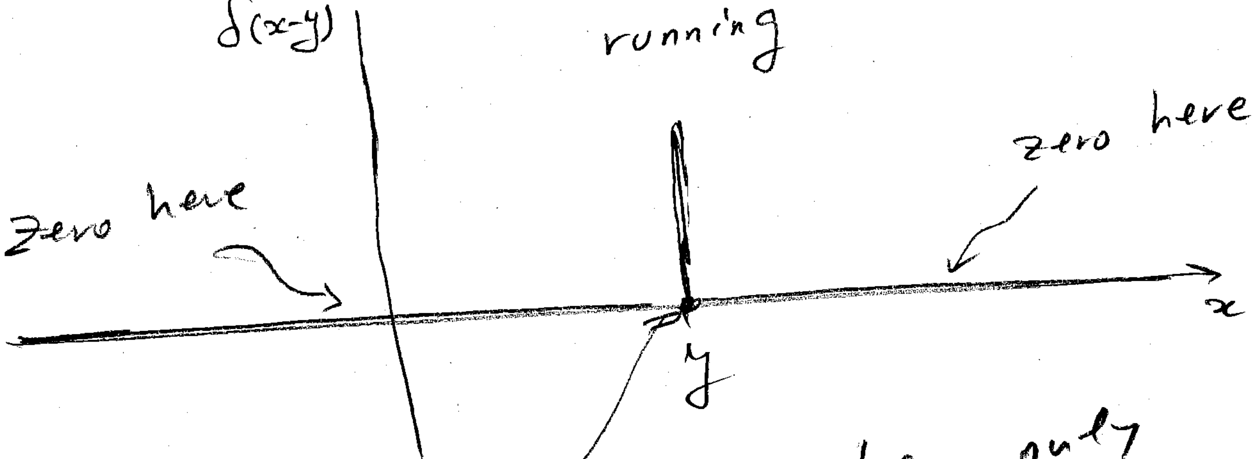
rewrite

Think of this j as being FIXED and i is running

The continuous version

$$\delta(x-y)$$

think of this y as being FIXED and x is running



a nonzero value only at y

The undifined value at y is a problem.
We need to understand the behavior at y.

The continuous version $\delta(x-y)$ is known as the Dirac's delta function.
We need to <u>transfer</u> properties of the discrete delta function to a continuous realm. Then the new object, Dirac's delta becomes useful.

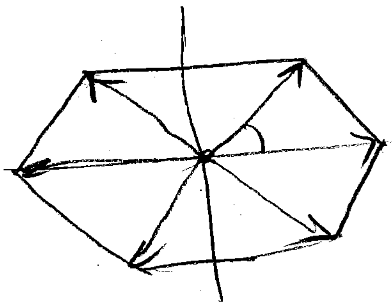
Discrete $\delta_{ij} = \delta(i-j)$	Continuous $\delta(x-y)$
$\sum_{i=0}^{N} \delta_{ij} = 0$ for every j	$\int_{-\infty}^{\infty} dx\, \delta(x-y) = 1$ for every y.
$\sum_{m=0}^{N} \delta_{mj} f_m = f_j$ FILTERING PROPERTY that is if you want to select (filter) on component f_j from $[f_1, f_2, \ldots, f_N]$, you need to multiply the vector with the FILTER δ_{mj} and sum. POSITION OF THE FILTER	$\int_{-\infty}^{\infty} dx\, \delta(x-y) f(x) = f(y)$ The FILTER $\delta(x-y)$ sits at y and selects only one component $f(y)$

From
$$\int_{-\infty}^{\infty} dx\, \delta(x-y) = 1$$

and the condition
$$\delta(x-y) = \begin{cases} 0 & x \neq y \\ \text{something} & x = y \end{cases}$$

we obtain that something $= \infty$, and even that it does not make sense if $\int_{-\infty}^{\infty}$ is an integral. Anyway, this small obstacle is will not stop us to be free to construct new symbols.

From simple geometry to δ-function



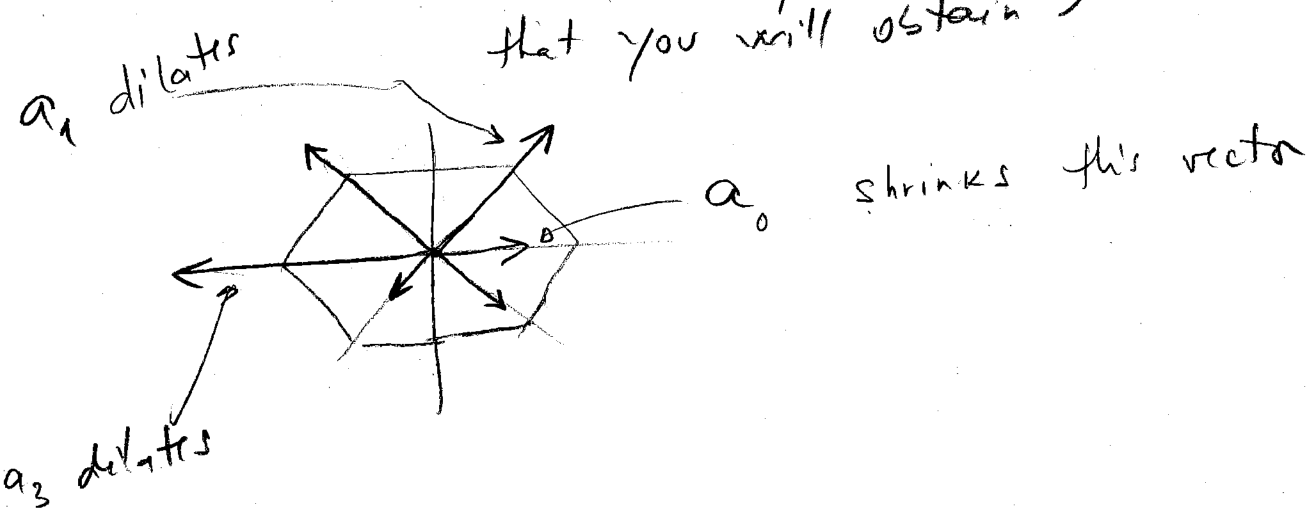
$\sum$ vectors $= 0$

$$\sum_{k=0}^{N-1} e^{-i\frac{2\pi}{N}\cdot k} = 0$$

Now if you distort the lengths of the vectors you can obtain any complex number

$$\sum_{k=0}^{N-1} a_k \, e^{-i\frac{2\pi}{N}\cdot k} = \text{A COMPLEX NUMBER THAT YOU NEED}$$

choose this (a_k) in such a way that you will obtain



Now make $-i\frac{2\pi}{N} = -2\pi\theta$

$$\sum_k a_k e^{-i\frac{2\pi}{N}k} \xrightarrow[\text{case}]{\text{continuous}} \int a(x) e^{-i2\pi\theta x} dx$$

So if you choose $a(x)$ and integrate you obtain a number which depends on θ.

$$\tilde{a}(\theta) = \int_{-\infty}^{\infty} a(x) e^{-i2\pi\theta x}$$

Going back to the discrete case

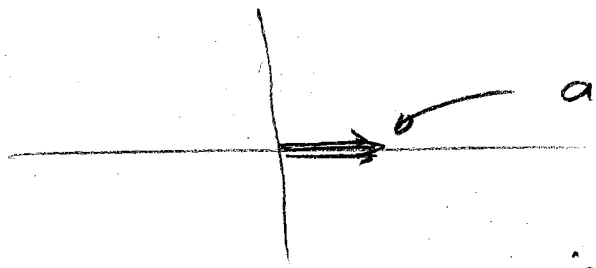
$$\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}k} = 0$$

the continuous case is (if $a(x) = 1$)

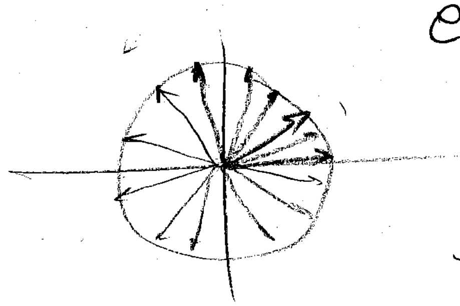
$$\int_{-\infty}^{\infty} e^{-i2\pi\theta x} dx = 0$$

But if $\theta = 0$ we get $\int_{-\infty}^{\infty} 1\, dx = \infty$!

So $$\int_{-\infty}^{\infty} e^{-i2\pi\theta x}\, dx = \begin{cases} 0 & \theta \neq 0 \\ \infty & \theta = 0 \end{cases}$$



all $e^{-i2\pi\theta x} = 1$ are aligned for $\theta = 0$



$e^{-i2\pi\theta x}$ are spread symmetrical for $\theta \neq 0$.

Here θ is FIXED and x is a continuous INDEX that label each vector.

Guess

$$\boxed{\int_{-\infty}^{\infty} e^{-i2\pi\theta x}\, dx = \delta(\theta)}$$

This is true; a "proof" is at the end of my writings.

Because $\delta(\theta) = \delta(-\theta)$ (symmetry)

$$\int_{-\infty}^{\infty} e^{i2\pi\theta x}\, dx = \delta(\theta)$$

Fourier Transform

$$\tilde{P}(k) = \int_{-\infty}^{\infty} dx \; e^{i 2\pi k x} \; P(x)$$

Define this $\tilde{P}(k)$.

Goal find $P(x)$ in terms of $\tilde{P}(k)$.

Solution

$$P(x) = \int_{-\infty}^{\infty} dy \; P(y) \, \delta(y-x) =$$

$$= \int_{-\infty}^{\infty} dy \; P(y) \int_{-\infty}^{\infty} dk \; e^{i 2\pi k (y-x)} =$$

$$\stackrel{*}{=} \int_{-\infty}^{\infty} dk \; e^{-i 2\pi k x} \int_{-\infty}^{\infty} dy \; P(y) \, e^{i 2\pi k y} =$$

$$= \int_{-\infty}^{\infty} dk \; e^{-i 2\pi k x} \; \tilde{P}(k)$$

interchange the integration and factor $e^{-i 2\pi k x}$ which does not depend on y

i.e., all these terms are much less than unity when n is large. Hence (A · 6 · 12) can be approximated by

$$n! = n^n e^{-n} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\xi^2/n)} \left[1 + \left(\frac{1}{3}\frac{\xi^3}{n^2} - \frac{1}{4}\frac{\xi^4}{n^3} \right) + \left(\frac{1}{18}\frac{\xi^6}{n^4} + \cdots \right) \right] d\xi \qquad \text{(A · 6 · 13)}$$

where we have used the expansion

$$e^y = 1 + y + \tfrac{1}{2}y^2 + \cdots$$

and have been careful to retain all terms of order $n^{-\frac{1}{2}}$ and n^{-1}. (Note that $\xi^6/n^4 \lesssim n^3/n^4 \approx n^{-1}$ is still of order n^{-1}.) Here the second integral involving ξ^3 vanishes by symmetry, since the integrand is an odd function of ξ. The remaining three integrals can be evaluated by the results of Appendix A · 4. Hence

$$\begin{aligned} n! &= n^n e^{-n} \left\{ \sqrt{2\pi n} + 0 - \frac{1}{4n^3}\left[\frac{3}{4}\sqrt{\pi}\,(2n)^{\frac{5}{2}} \right] + \frac{1}{18n^4}\left[\frac{15}{8}\sqrt{\pi}\,(2n)^{\frac{7}{2}} \right] \right\} \\ &= \sqrt{2\pi n}\, n^n e^{-n} \left[1 - \frac{3}{4n} + \frac{5}{6n} \right] \end{aligned}$$

Thus

$$n! = \sqrt{2\pi n}\, n^n e^{-n} \left[1 + \frac{1}{12n} + \cdots \right] \qquad \text{(A · 6 · 14)}$$

This shows the next correction term for Stirling's formula. Thus, even when n is as small as 10, Stirling's formula is already accurate to better than 1 percent.*

A · 7 *The Dirac delta function*

The Dirac δ function is a very convenient "function" (or more exactly, the limiting case of a function) having the property of singling out a particular value $x = x_0$ of a variable x. The function is characterized by the following properties:

$$\left. \begin{array}{lll} & \delta(x - x_0) = 0 & \text{for } x \neq x_0 \\ \text{but} & \delta(x - x_0) \to \infty & \text{for } x \to x_0 \\ \text{in such a way that,} & & \\ \text{for any } \epsilon > 0, & \displaystyle\int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x - x_0)\, dx = 1 & \end{array} \right\} \qquad \text{(A · 7 · 1)}$$

That is, the function $\delta(x - x_0)$ has a very sharp peak at $x = x_0$, but the area under the peak is unity. It follows that, given any smooth function $f(x)$, one

* More rigorous estimates of the maximum error committed in using Stirling's formula can be found in R. Courant, "Differential and Integral Calculus," p. 361, Interscience Publishers, New York, 1938; also in R. Courant and D. Hilbert, "Methods of Mathematical Physics, vol. I, p. 522, Interscience Publishers, New York, 1953.

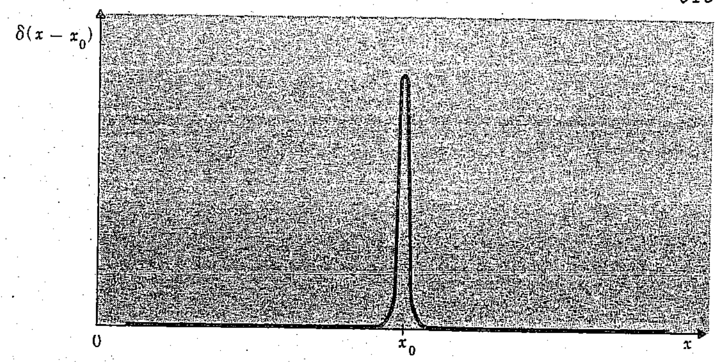


Fig. A·7·1 Schematic plot of $\delta(x - x_0)$ as a function of x.

has

$$\int_A^B f(x)\delta(x - x_0)\,dx = f(x_0)\int_A^B \delta(x - x_0)\,dx$$

since $\delta(x - x_0) \neq 0$ only when $x = x_0$, and there $f(x) = f(x_0)$. Hence

$$\int_A^B f(x)\delta(x - x_0)\,dx = \begin{cases} f(x_0) & \text{if } A < x_0 < B \\ 0 & \text{otherwise} \end{cases} \qquad (\text{A}\cdot 7\cdot 2)$$

The property (A·7·2) implies all the characteristics of (A·7·1) and can be taken as the definition of the function $\delta(x - x_0)$.

The δ function is a mathematical representation of a very common physical approximation, the "physical point." (An example is the electron considered as a point charge.) It corresponds to a finite physical quantity (e.g., an electrical charge) concentrated in a region much smaller than all other dimensions relevant in a physical discussion. Subtle questions concerned with limiting processes involving mathematical points are therefore usually irrelevant in discussions of physical problems.*

The following are examples of various analytical representations of the δ function. In all of these the positive parameter γ is taken in the limit $\gamma \to 0$. (This is a physical limit where γ is smaller than all other relevant dimensions.)

Example 1: $$\delta(x) = \begin{cases} \dfrac{1}{\gamma} & \text{for } -\dfrac{\gamma}{2} < x < \dfrac{\gamma}{2} \\ 0 & \text{otherwise} \end{cases} \qquad (\text{A}\cdot 7\cdot 3)$$

Example 2: $$\delta(x) = \frac{1}{\pi}\frac{\gamma}{x^2 + \gamma^2} \qquad (\text{A}\cdot 7\cdot 4)$$

Example 3: $$\delta(x) = \frac{1}{\sqrt{2\pi}\,\gamma}e^{-x^2/2\gamma^2} \qquad (\text{A}\cdot 7\cdot 5)$$

The most convenient and important representation is, however, one involving an integral.

* The reader interested primarily in questions of mathematical rigor is referred to M. J. Lighthill, "Introduction to Fourier Analysis and Generalized Functions," Cambridge University Press, Cambridge, 1958.

Integral representation of the δ function The periodic character of the complex exponential function yields the familiar result

$$\int_{-\pi}^{\pi} e^{in\phi}\,d\phi = \begin{cases} 2\pi & \text{for } n = 0 \\ \dfrac{e^{in\pi} - e^{-in\pi}}{in} = \dfrac{[(\pm 1) - (\pm 1)]}{in} = 0 & \text{for } n \neq 0 \end{cases}$$

i.e.,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\phi}\,d\phi = \delta_{n,0} \tag{A·7·6}$$

where the right-hand side is a shorthand notation defined by

$$\delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \tag{A·7·7}$$

This useful symbol is called the "Kronecker delta symbol." It is obviously the analog, for discrete variables, of the definition of the Dirac δ function $\delta(x - x_0)$ for continuous variables.

To make the connection between the discrete and continuous cases explicit, choose a very large number L so that

$$x \equiv \frac{2\pi n}{L} \tag{A·7·8}$$

covers essentially all possible values of the continuous variable x as n assumes all possible integral values.* The relation (A·7·8) associates with each integer n the range of x lying between

$$\frac{2\pi}{L}\left(n - \frac{1}{2}\right) < x < \frac{2\pi}{L}\left(n + \frac{1}{2}\right)$$

i.e., a range of magnitude

$$\Delta x = \frac{2\pi}{L} \tag{A·7·9}$$

which becomes infinitesimally small as $L \to \infty$.

* The factor 2π in (A·7·8) is introduced purely for convenience so that trigonometric functions such as $\cos Nx$ (where N is any integer) remain unchanged when x changes by L.

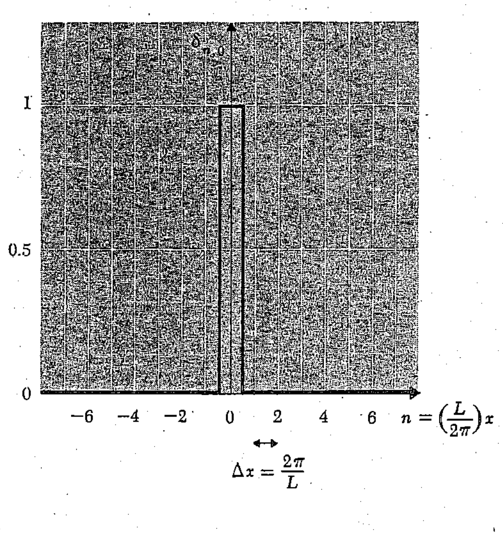


Fig. A·7·2 The Kronecker delta symbol $\delta_{n,0}$ as a function of the continuous variable $x = 2\pi n/L$.

According to the definition (A·7·7) it then follows that

$$\delta_{n,0} = \begin{cases} 1 & \text{when } -\frac{1}{2}\,\Delta x < x < \frac{1}{2}\,\Delta x \\ 0 & \text{otherwise} \end{cases}$$

By (A·7·3) one can then write for the δ function the expression

$$\delta(x) = \lim_{\Delta x \to 0} \frac{\delta_{n,0}}{\Delta x} \tag{A·7·10}$$

Hence $\quad \delta(x) = \lim_{\Delta x \to 0} \dfrac{1}{2\pi\,\Delta x} \displaystyle\int_{-\pi}^{\pi} e^{in\phi}\,d\phi \qquad$ using (A·7·6)

$$= \lim_{L \to \infty} \frac{1}{2\pi}\left(\frac{L}{2\pi}\right) \int_{-\pi}^{\pi} e^{(iLx/2\pi)\phi}\,d\phi \qquad \text{using (A·7·8) and (A·7·9)}$$

Let
$$k \equiv \frac{L}{2\pi}\,\phi$$

so that
$$d\phi = \frac{2\pi}{L}\,dk$$

Then

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\,dk \tag{A·7·11}$$

This is the desired integral representation of the δ function, a very useful result.

Remark Let us verify directly that the representation (A·7·11) has the desired properties of the δ function. We see that for $x \neq 0$, the integrand in (A·7·11) is a rapidly oscillating function so that the integral vanishes; also for $x = 0$, $e^{ikx} = 1$, so that the integral approaches infinity. In more detail, introduce in the integrand the factor $e^{-\gamma|k|}$ (where γ is infinitesimally small and positive) to avoid convergence ambiguities for $|k| \to \infty$. Then

$$\begin{aligned}
\delta(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\gamma|k|}\,dk \\
&= \frac{1}{2\pi}\int_0^{\infty} e^{ixk-\gamma k}\,dk + \frac{1}{2\pi}\int_{-\infty}^{0} e^{ixk+\gamma k}\,dk \\
&= \frac{1}{2\pi}\int_0^{\infty} e^{(ix-\gamma)k}\,dk + \frac{1}{2\pi}\int_{-\infty}^{0} e^{(ix+\gamma)k}\,dk \\
&= \frac{1}{2\pi}\left\{\frac{[e^{(ix-\gamma)k}]_0^{\infty}}{ix-\gamma} + \frac{[e^{(ix+\gamma)k}]_{-\infty}^{0}}{ix+\gamma}\right\} \\
&= \frac{1}{2\pi}\left\{\frac{-1}{ix-\gamma} + \frac{1}{ix+\gamma}\right\}
\end{aligned}$$

or
$$\delta(x) = \frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}, \qquad \text{where } \gamma \to 0 \tag{A·7·12}$$

This is the same as the representation (A·7·4) mentioned previously. Note that, for $x \neq 0$, $\delta(x) = \gamma/\pi x^2 \to 0$; for $x = 0$, $\delta(x) = (\pi\gamma)^{-1} \to \infty$. Also the integral of (A·7·12) is

$$\int_{-\infty}^{\infty} \frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}\,dx = \frac{1}{\pi}\left[\tan^{-1}\frac{x}{\gamma}\right]_{-\infty}^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

as required.

Thus (A · 7 · 6) and (A · 7 · 11) yield the very useful results

$$\delta_{n,m} = \delta_{m,n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\phi(n-m)}\,d\phi \tag{A·7·13}$$

$$\delta(x - x_0) = \delta(x_0 - x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_0)}\,dk \tag{A·7·14}$$

Remark Since (A · 7 · 13) vanishes when $n \neq m$, its right side can, without affecting the equality, be multiplied by any function which reduces to unity when $n = m$. In particular, one can write the more general result

$$\delta_{n,m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\phi(n-m)}\,d\phi \cdot e^{\phi_0(n-m)}$$

or

$$\delta_{n,m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(\phi_0+i\phi)(n-m)}\,d\phi \tag{A·7·15}$$

where ϕ_0 is *any* arbitrary parameter. Similarly, one can write

$$\delta(x - x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(k_0+ik)(x-x_0)}\,dk \tag{A·7·16}$$

where k_0 is *any* arbitrary parameter.

A · 8 *The inequality* $\ln x \leq x - 1$

We wish to compare $\ln x$ with x itself for positive values of x. Consider the difference function

$$f(x) \equiv x - \ln x \tag{A·8·1}$$

$$\left.\begin{array}{lll} \text{For } x \to 0, & \ln x \to -\infty; & \text{hence } f(x) \to \infty \\ \text{For } x \to \infty, & \ln x \ll x; & \text{hence } f(x) \to \infty \end{array}\right\} \tag{A·8·2}$$

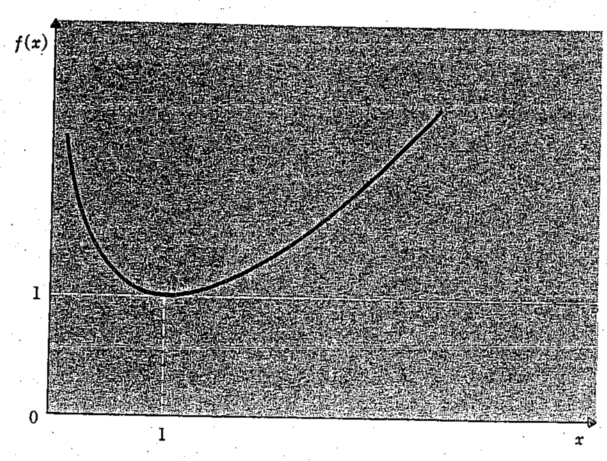


Fig. A·8·1 *The function $f(x) \equiv x - \ln x$ as a function of x.*