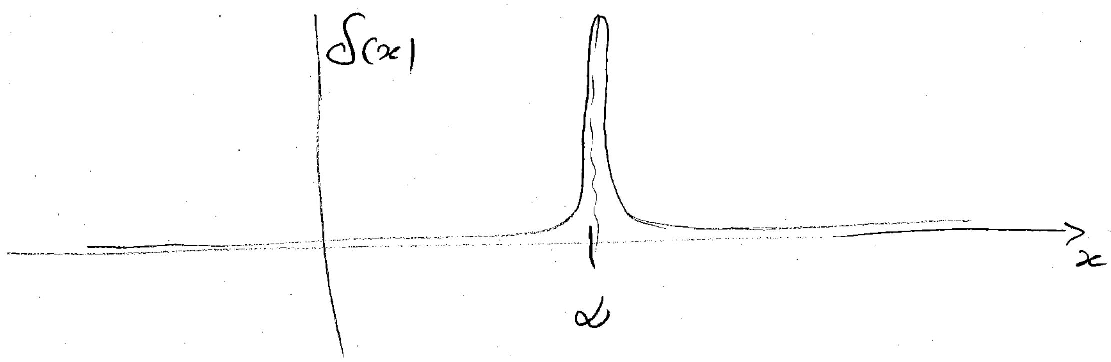
About Dirac's δ function

$$\delta(x-\alpha)=0 \quad \text{for } x \neq \alpha$$

$$\int_{-\infty}^{\infty} \delta(x-\alpha)\, f(x)\, dx = f(\alpha)$$



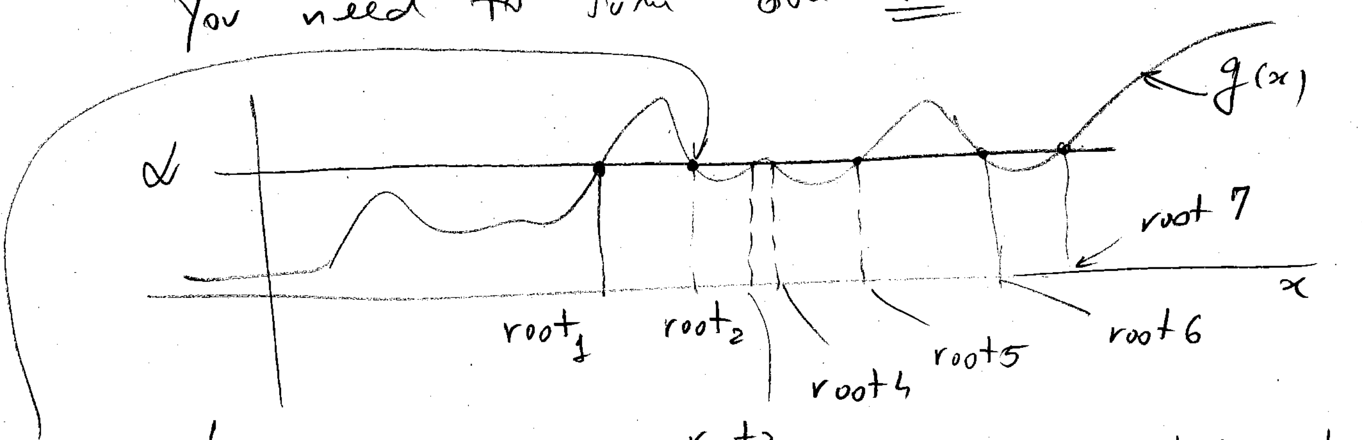
An important property

$$\int_{-\infty}^{\infty} \delta(g(x)-\alpha)\, f(x)\, dx = \sum_{\text{all roots}} \frac{f(\text{root})}{|g'(\text{root})|}$$

roots of the equation

$$g(x)=\alpha$$

You need to sum over all the roots



Here $g'(\text{root}_2)$ is negative so you take the absolute value $|g'(\text{root}_2)|$

Proof

1) $g(x) = ax$ with $a>0$

$$\int_{-\infty}^{\infty} \delta(ax-\alpha)\, f(x)\, dx =$$

$\boxed{\text{change to } y = ax}$

$$= \int_{-\infty}^{\infty} \delta(y-\alpha)\, f\left(\frac{y}{a}\right) \cdot \frac{dy}{a} =$$

$$= \frac{f\left(\frac{\alpha}{a}\right)}{a} = \frac{f(\text{root})}{g'(\text{root})}$$

where from $g(x) = \alpha$ you get

$$\text{root} = \frac{\alpha}{a}$$

2) $g(x) = -ax$ with $a>0$

$$\int_{-\infty}^{\infty} \delta(-ax-\alpha)\, f(x)\, dx = \int_{+\infty}^{-\infty} \delta(y-\alpha) f\left(\frac{y}{-a}\right) \frac{dy}{-a} =$$

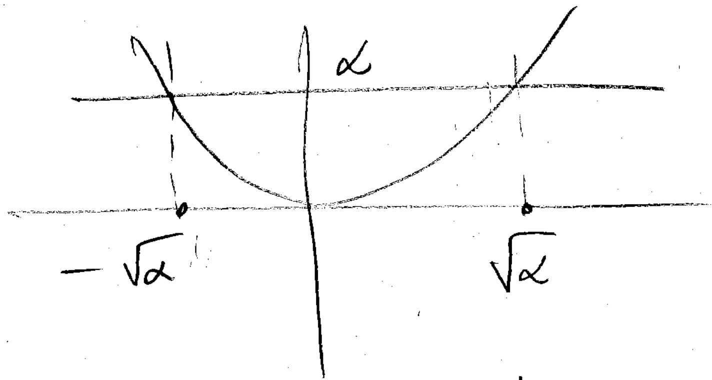
because $y = -ax$

change the limits of the integral

$$= -\int_{-\infty}^{\infty} \delta(y-\alpha)\, f\left(\frac{y}{-a}\right) \frac{dy}{-a} = \frac{f\left(\frac{\alpha}{-a}\right)}{a} =$$

$$= \frac{f(\text{root})}{|g'(\text{root})|}, \quad \text{root} = -\frac{\alpha}{a}$$

3) $g(x) = x^2$



2 roots

$$\int_{-\infty}^{\infty} \delta(x^2-\alpha) f(x)\, dx = \int_{-\infty}^{0} \delta(x^2-\alpha) f(x)\, dx + \int_{0}^{\infty} \delta(x^2-\alpha) f(x)\, dx =$$

[change the variable $y = x^2$ and follow the same prescription as in 1) and 2)] =

$$= \underbrace{\frac{f(-\sqrt{\alpha})}{|g'(-\sqrt{\alpha})|}}_{\text{from the first integral}} + \underbrace{\frac{f(\sqrt{\alpha})}{|g'(\sqrt{\alpha})|}}_{\text{from the second integral}}$$

<u>IDEA</u> : integrate around each root and then get a sum over all roots.

You can write the formula

$$\int_{-\infty}^{\infty} \delta(g(x)-\alpha) f(x)\, dx = \sum_{\substack{\text{all roots} \\ \text{of } g(x)=\alpha}} \frac{f(\text{root})}{|g'(\text{root})|}$$

as

$$\delta(g(x)-\alpha) = \sum_{\substack{\text{all roots} \\ \text{of } g(x)=\alpha}}' \frac{\delta(x-\text{root})}{|g'(\text{root})|}$$

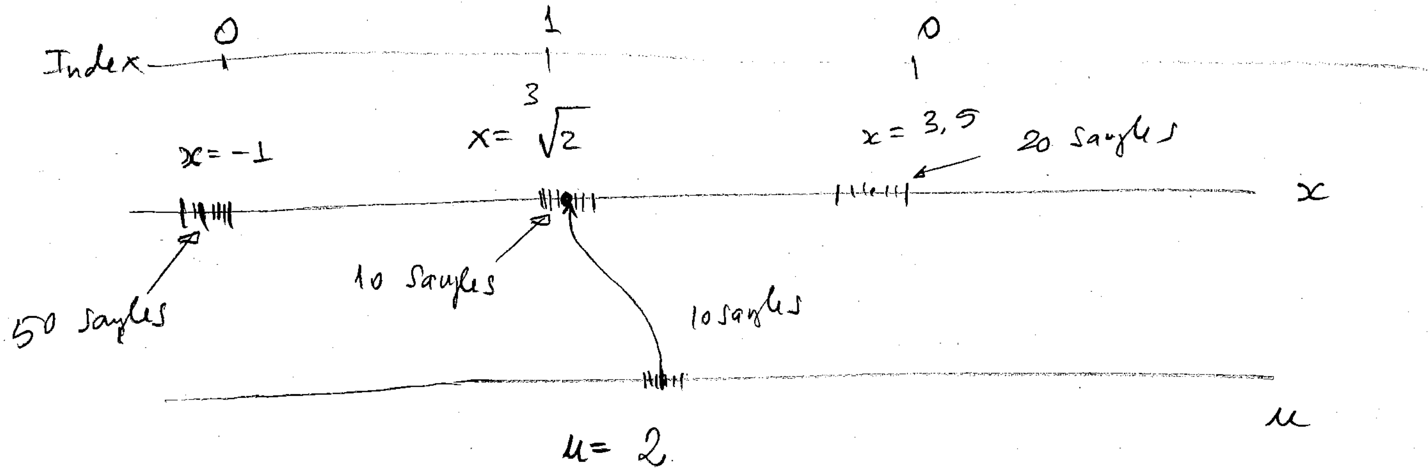
check it !

One more property

$$\delta(x) = \int_{-\infty}^{\infty} e^{-2\pi i k x}\, dk$$

many properties are at

http://mathworld.wolfram.com/DeltaFunction.html



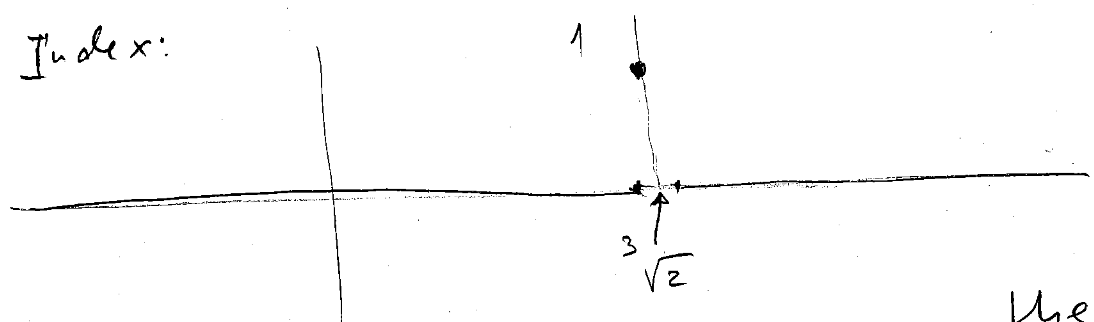
$0 \cdot 50 + \dots + 1 \cdot 10$ samples $+ \dots + 0 \cdot 20$ samples $=$

$= 10$ samples

Multiply with 0 everywhere except at one point where you multiply with 1

$\sum_{\text{over all } x}$ Index $\cdot$ # in x bins $=$ # in u bins

The Index:



For a fixed u (like $u=2$) the

Index $= \delta(u - x^3)$

check $\delta(2 - (\sqrt[3]{2})^3) = \delta(0)$ which is not zero!

# of samples in the x bin is $P(x)\,dx$

So

$$\sum_{\substack{\text{over all}\\ x}}' \text{Index} \cdot \#\text{ samples in } x \text{ bins} =$$

$$= \int_{-\infty}^{\infty} \delta(u - x^3)\, P(x)\, dx$$

So the probability density in u variable is

$$\boxed{P(u) = \int_{-\infty}^{\infty} \delta(u - x^3)\, P(x)\, dx}$$