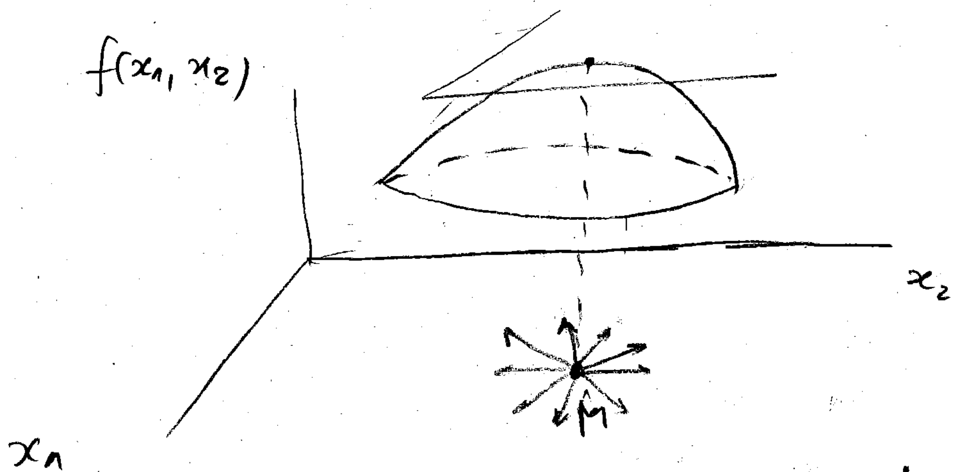
Lagrange multipliers

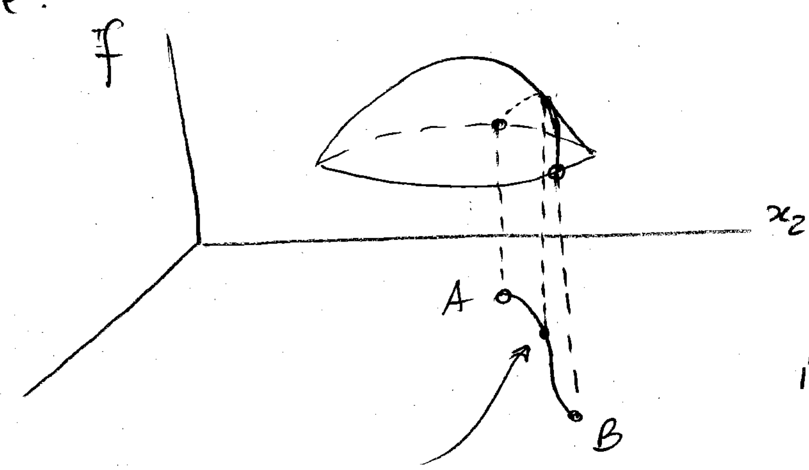
Maximum of $f(x_1, x_2)$



any small displacement out of the point M will keep F constant. thus the condition for M is

$$\left.\frac{\partial f}{\partial x_1}\right|_M = 0, \quad \left.\frac{\partial f}{\partial x_2}\right|_M = 0$$

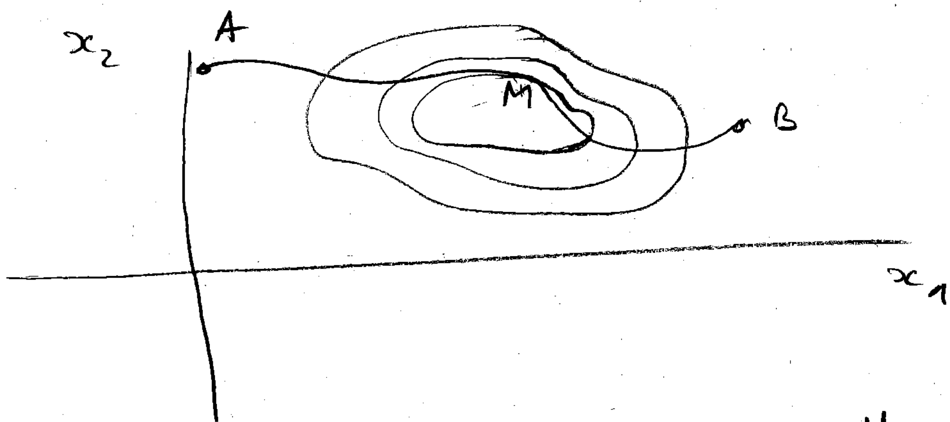
Now a constrain, you cannot move freely in the (x_1, x_2) plane.



A point on AB where F is maximum

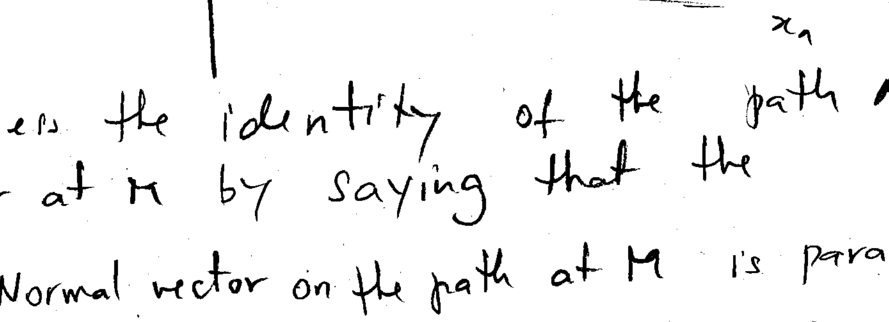
The path AB is given as a constrain $g(x_1, x_2) = 0$

Plot $f(x_1, x_2)$ as contour levels, to bring the plot close to the plot of the path AB, so it will be easier to analyze the problem



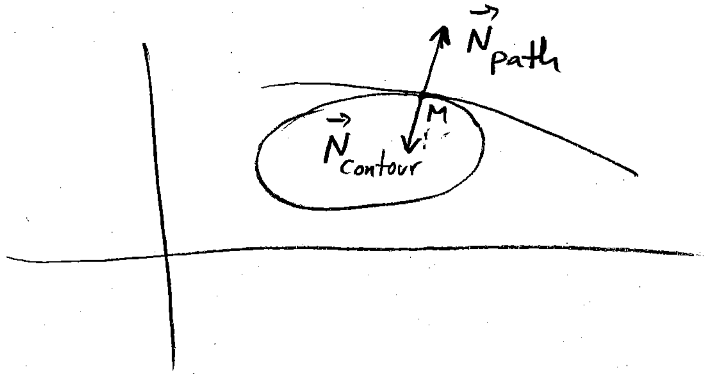
As we walk on the constraint path from A to B we will find that at the point M the value of $f(x_1, x_2)$ does not change in the vecinity of M. This is because in the vecinity of M the path AB and the contour of $f(x_1, x_2)$ are identical

M is a stationary point.



We express the identity of the path AB and the contour at M by saying that the

Normal vector on the path at M is parallel with the Normal vector on the contour at M.



$$\vec{N}_{contour} \parallel \vec{N}_{path}$$

So there must be a number λ such that

$$\vec{N}_{contour} = \lambda \vec{N}_{path}$$

this number is unknown to us and we have to find it.

On the other side

$$\vec{N}_{contour} = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right)$$

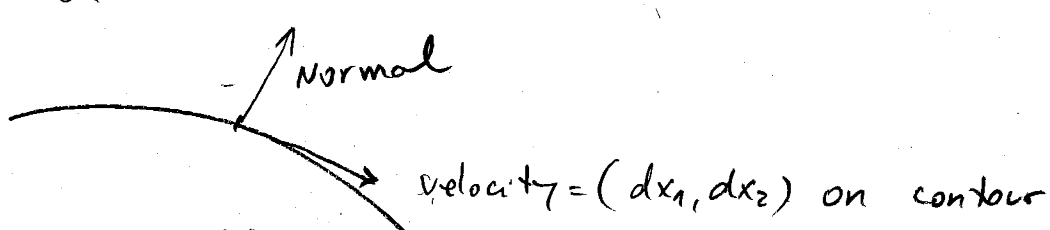
$$\vec{N}_{path} = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2} \right)$$

Why this ↗.

Because, in general

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$$

BUT, on contour $f = \text{CONSTANT}$ so $df|_{contour} = 0$



$$0 = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right) \cdot (dx_1, dx_2)$$

dot product

So the condition

$$\vec{N}_{contour} = \lambda \vec{N}_{path}$$

is

$$\frac{\partial f}{\partial x_1} = \lambda \frac{\partial g}{\partial x_1} \qquad (1)$$

$$\frac{\partial f}{\partial x_2} = \lambda \frac{\partial g}{\partial x_2} \qquad (2)$$

which is a FREE OF CONSTRAIN condition of stationarity for the function

$$\underline{f - \lambda g}$$

Conclusion and Lagrange idea

Given f and the constraint g, take an unknown number λ, multiply it with g (so the name Lagange multiplier) and form

$$f - \lambda g$$

Find the stationary points of $f - \lambda g$ as if no constraints are present. The solution to (1) and (2) will give you BOTH the (x_1, x_2) for the extremum and the number λ.

Method

Solve

$$\frac{\partial f}{\partial x_1} = \lambda \frac{\partial g}{\partial x_1} \qquad (1)$$

$$\frac{\partial f}{\partial x_2} = \lambda \frac{\partial g}{\partial x_2} \qquad (2)$$

and find the solution in terms of λ

$$\overline{x}_1(\lambda), \quad \overline{x}_2(\lambda) \qquad (3)$$

Then plug the solution into the constraint

$$g(\overline{x}_1(\lambda), \overline{x}_2(\lambda)) = 0 \qquad (4)$$

This will be an equation for λ.

Find $\hat{\lambda}$ the solution of (4)

$$\lambda = \hat{\lambda} \qquad (5)$$

Itroduce $\hat{\lambda}$ into (3) an get the stationary point

$$\overline{x}_1(\hat{\lambda}), \quad \overline{x}_2(\hat{\lambda})$$

In general, to obtain the stationary points of a function of m variables

$$f(x_1, x_2, \dots, x_m) \tag{6}$$

subjected to r constraints

$$\begin{cases} g_1(x_1, x_2, \dots, x_m) = 0 \\ \vdots \\ g_r(x_1, x_2, \dots, x_m) = 0 \end{cases} \tag{7}$$

Construct the function

$$F = f - \lambda_1 g_1 - \lambda_2 g_2 - \dots - \lambda_r g_r \tag{8}$$

Then solve the system of equations for

$$\frac{\partial F}{\partial x_1} = 0, \dots, \frac{\partial F}{\partial x_m} = 0 \tag{9}$$

and find

$$\begin{cases} \bar{x}_1(\lambda_1, \dots, \lambda_r) \\ \vdots \\ \bar{x}_m(\lambda_1, \dots, \lambda_r) \end{cases} \tag{10}$$

Introduce (10) into the constraints (7) and solve for $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_r$. Finally, the stationary point is, from (10)

$$\begin{cases} \bar{x}_1(\hat{\lambda}_1, \dots, \hat{\lambda}_r) \\ \vdots \\ \bar{x}_m(\hat{\lambda}_1, \dots, \hat{\lambda}_r) \end{cases} \tag{11}$$